Physica D 321–322 (2016) 51–66

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Numerical simulation of surface waves instability on a homogeneous grid

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HIGHLIGHTS

- We described three and four wave instabilities due to resonant interactions.
- Algorithm for simulation of weakly nonlinear surface waves is presented.
- Numerical scheme conserves Hamiltonian of the system.
- We discussed and simulated instability of standing and propagating waves.

ARTICLE INFO

Article history:
Received 28 August 2015
Accepted 27 February 2016
Available online 10 March 2016
Communicated by M. Vergassola

Keywords:
Water waves
Numerical simulation
Weak turbulence

ABSTRACT

We performed full-scale numerical simulation of instability of weakly nonlinear waves on the surface of deep fluid. We show that the instability development leads to chaotization and formation of wave turbulence.

Instability of both propagating and standing waves was studied. We separately studied pure capillary wave, that was unstable due to three-wave interactions and pure gravity waves, that were unstable due to four-wave interactions. The theoretical description of instabilities in all cases is included in the article. The numerical algorithm used in these and many other previous simulations performed by the authors is described in detail.

1. Introduction

Stationary propagating waves on the surface of deep, heavy ideal fluid have been known since the middle of the nineteenth century. Stokes (see, for instance [1]) in 1847 found the solution of the Euler equation in the form of trigonometric series. For the shape of surface $\eta(x,t)$, he obtained:

$$\eta(x,t) = a\left[\cos(kx-\omega t) + \frac{1}{2}\mu\cos\{2(kx-\omega t)\} + \frac{3}{8}\mu^2\cos\{3(kx-\omega t)\} + \cdots\right]. \quad (1)$$

Here we introduced steepness μ and frequency ω

$$\mu = ka, \qquad \omega = \sqrt{gk}\left(1 + \frac{1}{2}\mu^2 + \frac{1}{8}\mu^4 + \cdots\right). \quad (2)$$

Stokes found two algorithms for the calculation of all terms in series (1) and (2) (see Sretenskii [2]). Convergence of these series was proven by Nekrasov [3,4] in 1921. Another proof was found by Levi-Civita [5]. Recently shapes of different Stokes waves were obtained numerically with high precision [6,7], their analytic structure was revealed [7] and explained [8].

It has been known since 1965 [9] that stationary waves on the surface of deep water are unstable. The theory of instability [10–13] was developed for waves of small amplitude within the limit $\mu \to 0$. A history of this question is described in the article [14]. Recent advances can be found in [15]. In the present paper, we study the instability of stationary waves numerically through the direct solution of the Euler equation which describes a

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http://dx.doi.org/10.1016/j.physd.2016.02.017
0167-2789/

potential flow of ideal fluid with free surface. This approach has two important advantages. Firstly, through numerical simulation we can study waves with finite amplitudes. While this paper focuses only on cases of small amplitude, such an advantage will be crucial for other applications, e.g. wave breaking simulation. Secondly, the use of numerical simulation allows us to study not only linear, but also nonlinear stages of instability development. Even in integrable systems like the NLSE, analytical study of the monochromatic wave is a very nonlinear problem and can be solved only by methods of algebraic geometry [16]. In more realistic models, development of a nonlinear theory of modulational instability for waves is a hopeless problem. In the long run, we have to expect that instability will lead to the formation of a stochastic wave field described by a kinetic equation for squared wave amplitudes and formation of Kolmogorov–Zakharov (KZ) spectra, governed by the energy flux in high wave numbers [17].

The article is organized as follows. Sections 2 and 3 are devoted to analytical theory of stability of weakly nonlinear stationary waves. To develop this theory, we use Hamiltonian formalism as this approach is the most compact and suitable. We start with presenting the Euler equation of ideal fluid with free surface in the Hamiltonian form. Surface tension is also included in the Hamiltonian. In the presence of surface tension, the dispersion relation is:

$$\omega_k = \sqrt{gk + \sigma k^3},$$

where σ—the surface tension coefficient (here and further we consider fluid of unit density).

Wave vectors of small-amplitude stationary waves are solutions of the equation

$$\omega_k = ck. \tag{3}$$

This equation has two solutions (we omit trivial solution $k = 0$):

$$k_{1,2} = \frac{c^2 \pm \sqrt{c^4 - 4g\sigma}}{2\sigma}, \tag{4}$$

if $c > c_0$, where $c_0 = (4g\sigma)^{1/4}$. For water, $c_0 \simeq 23$ cm/s. In a generic case $c \sim c_0$ stationary waves comprise a complicated four-parameter family. However, in the limiting case $c \gg c_0$ one can split it into two periodic families of "pure gravitational" and "pure capillary" waves.

The Stokes wave is "pure gravitational". Now, with $k_1 = g/c^2$ capillary effects can be neglected. In the "pure capillary" case $k_2 = c^2/\sigma$, effects of gravity can be ignored. All stationary waves on the surface of deep fluid are unstable. However, the instabilities of short capillary waves and long gravity waves are significantly different and described by different "efficient Hamiltonians". The case of "pure capillary" waves is the simplest. The instability can be studied if Hamiltonian contains only quadratic and cubic terms. This is the subject of Section 2. A situation is more complicated for gravitational waves. In this case, fourth order terms must be included in the Hamiltonian. Then, one has to exclude the cubic terms through a proper conformal transformation. As a result, we get so-called "Zakharov equation" [13]. In the framework of this equation, the problem of the Stokes wave stability can be solved exactly. This is the subject of Section 3.

In Section 4, we give a detailed description of the numerical code which we used for the solution of the Hamiltonian Euler equation. This code was used in many papers but was never described in detail [18–25]. We should stress that in our numerical experiments we worked with the Euler equation written in "natural variables". These equations are not as good for the direct analytical study as they are good for the implementation of numerical method. The structure of nonlinear parts of the Hamiltonian in "natural variables" is relatively simple, and numerical implementation through standard Fast Fourier Transform (FFT) is quite feasible.

In Section 5 we present our results on the modeling of capillary wave instability. We show that an initial stage of instability is described pretty well by the linear analytical theory. Further development of instability leads to the appearance of "secondary instabilities" and a tendency toward the formation of a chaotic wave field, which should be described by statistical methods.

In Section 6, we study the instability of the Stokes wave. We show that this instability is mostly "modulational". In other words, the wave remains quasi-monochromatic for a long time after the development of the instability.

Finally, in Section 7 we present first results on the development of the standing wave instability. We show that this instability leads to fast isotropization of the wave field. This mechanism can be used in experiments for generation of an isotropic wave field.

2. Theory of decay instability

In this section, we develop the simplest version of the theory of stationary wave instability. This simple theory is applicable if triple-wave nonlinear processes governed by the resonant conditions

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \qquad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \tag{5}$$

are permitted. Let us briefly describe how the theory of surface waves can be embedded into the general Hamiltonian theory of nonlinear waves, before we use conditions (5).

Suppose that ideal incompressible fluid fills the space $-\infty < z < \eta(\mathbf{r}, t)$, here $\mathbf{r} = (x, y)$—two dimensional vector. A flow is potential $\mathbf{v} = \nabla\Phi$, hence hydrodynamical potential Φ satisfies the Laplace equation

$$\nabla^2\Phi = 0. \tag{6}$$

Let us define $\psi = \Phi|_{z=\eta}$ and impose a natural boundary condition $\Phi_z \to 0$ at $z \to -\infty$. It is known [10] that $\eta(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ are canonically conjugated variables satisfying evolutionary equations

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{7}$$

Here $H = T + U$—total energy of the fluid, consisting of kinetic energy

$$T = \frac{1}{2}\int \mathrm{d}^2\mathbf{r} \int_{-\infty}^{\eta} (\nabla\Phi)^2 \mathrm{d}z, \tag{8}$$

and potential energy

$$U = \frac{g}{2}\int \eta^2 \mathrm{d}^2\mathbf{r} + \sigma\int \left(\sqrt{1 + (\nabla\eta)^2} - 1\right)\mathrm{d}^2\mathbf{r}. \tag{9}$$

The Hamiltonian H in terms of η and ψ is given by the infinite series

$$H = H_0 + H_1 + H_2 + \cdots. \tag{10}$$

Here

$$H_0 = \frac{1}{2}\int \left\{\psi\hat{k}\psi + g\eta^2 + \sigma(\nabla\eta)^2\right\} \mathrm{d}^2\mathbf{r}, \tag{11}$$

here $\hat{k}\psi = \sqrt{-\nabla^2}\psi$,

$$H_1 = \frac{1}{2}\int \eta\{|\nabla\psi|^2 - (\hat{k}\psi)^2\}\mathrm{d}^2\mathbf{r}, \tag{12}$$

$$H_2 = \frac{1}{2}\int \eta(\hat{k}\psi)[\hat{k}(\eta\hat{k}\psi) + \eta\nabla^2\psi]\mathrm{d}^2\mathbf{r} + \frac{1}{2}\sigma\int (\nabla\eta^2)^2\mathrm{d}^2\mathbf{r}. \tag{13}$$

Thereafter, we will neglect the last term in (13).