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Partial classification of Lorenz knots: Syllable permutations of torus knots words

Paulo Gomes [a,*], Nuno Franco [b], Luís Silva [c]

[a] Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa, Portugal
[b] CIMA-UE and Departamento de Matemática, Universidade de Évora, Portugal
[c] CIMA-UE and Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa, Portugal

HIGHLIGHTS

- We define syllable permutations of symbolic words of Lorenz torus knots.
- We build an algorithm to construct symbolic words of satellite Lorenz knots.
- Some of the syllable permutation families of words are shown to be hyperbolic.
- Infinite families of hyperbolic Lorenz knots are generated in this way.
- The techniques used can be generalized to other families of Lorenz knots.

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ABSTRACT

We define families of aperiodic words associated to Lorenz knots that arise naturally as syllable permutations of symbolic words corresponding to torus knots. An algorithm to construct symbolic words of satellite Lorenz knots is defined. We prove, subject to the validity of a previous conjecture, that Lorenz knots coded by some of these families of words are hyperbolic, by showing that they are neither satellites nor torus knots and making use of Thurston's theorem. Infinite families of hyperbolic Lorenz knots are generated in this way, to our knowledge, for the first time. The techniques used can be generalized to study other families of Lorenz knots.

1. Introduction

Lorenz knots

Lorenz knots are the closed (periodic) orbits in the Lorenz system [1]

$$x' = -10x + 10y$$
$$y' = 28x - y - xz \qquad (1)$$
$$z' = -\frac{8}{3}z + xy$$

while *Lorenz links* are finite collections of (possibly linked) Lorenz knots.

The systematic study of Lorenz knots and links was made possible by the introduction of the *Lorenz template* or knot-holder by Williams in [2,3]. It is a branched 2-manifold equipped with an expanding semi-flow, represented in Fig. 1. It was first conjectured by Guckenheimer and Williams and later proved through the work of Tucker [4] that every knot and link in the Lorenz system can be projected into the Lorenz template. Birman and Williams made use of this result to investigate Lorenz knots and links [5]. For a review on Lorenz knots and links, see also [6].

A $T(p, q)$ torus knot is (isotopic to) a curve on the surface of an unknotted torus T^2 that intersects a meridian p times and a longitude q times. Birman and Williams [5] proved that every torus knot is a Lorenz knot.

A satellite knot is defined as follows: take a nontrivial knot C (companion) and nontrivial knot P (pattern) contained in a solid unknotted torus T and not contained in a 3-*ball* in T. A satellite knot is the image of P under an homeomorphism that takes the core of T onto C.

* Corresponding author.
E-mail addresses: pgomes@adm.isel.pt (P. Gomes), nmf@uevora.pt (N. Franco), lfs@adm.isel.pt (L. Silva).

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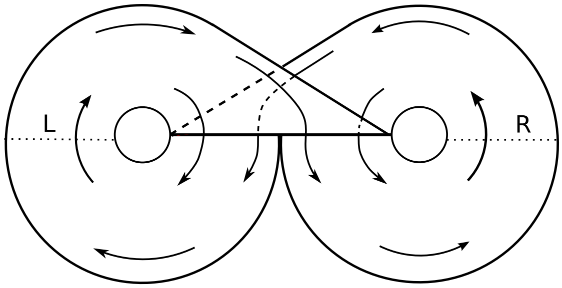

Fig. 1. The Lorenz template.

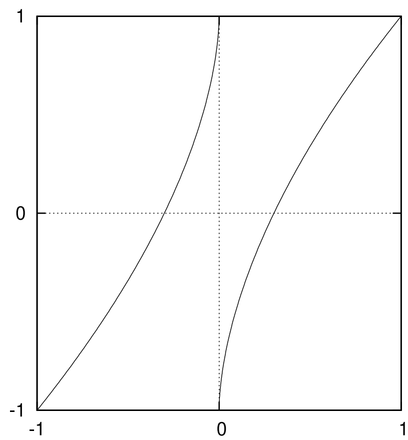

Fig. 2. Lorenz map.

A knot is hyperbolic if its complement in S^3 is a hyperbolic 3-*manifold*. Thurston [7] proved that a knot is hyperbolic *iff* it is neither a satellite knot nor a torus knot. One of the goals in the study of Lorenz knots has been their classification into *hyperbolic* and *non-hyperbolic*, possibly further distinguishing torus knots from satellites. Birman and Kofman [8] listed hyperbolic Lorenz knots taken from a list of the simplest hyperbolic knots. In a previous article we generated and tested for hyperbolicity, using the program *SnapPy*, families of Lorenz knots that are a generalization of some of those that appear in this list, which led us to conjecture that the families tested are hyperbolic [9].

The first-return map induced by the semi-flow on the *branch line* (the horizontal line in Fig. 1) is called the *Lorenz map*. If the branch line is mapped onto $[-1, 1]$, then the Lorenz map f becomes a one-dimensional map from $[-1, 1] \setminus \{0\}$ onto $[-1, 1]$, with one discontinuity at 0 and strictly increasing in each of the subintervals $[-1, 0[$ and $]0, 1]$ (Fig. 2).

Lorenz braids

If the Lorenz template is cut open along the dotted lines in Fig. 1, then each knot and link on the template can be obtained as the closure of an open braid on the cut-open template, which will be called the *Lorenz braid* associated to the knot or link [5]. These *Lorenz braids* are simple positive braids (our definition of positive crossing follows Birman and is therefore opposed to an usual convention in knot theory). Each Lorenz braid is composed of $n = p + q$ strings, where the set of p left or L strings cross over at least one (possibly all) of the q right strings, with no crossings between strings in each subset. These sets can be subdivided into subsets LL, LR, RL and RR according to the position of the startpoints and endpoints of each string. An example of a Lorenz braid is shown in Fig. 3, where we adopt the convention of drawing the overcrossing (L) strings as thicker lines than the undercrossing (R) strings. This convention will be used in other braid diagrams.

The *braid group on n strings* B_n is given by the presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (i = 1, \ldots, n-2) \end{array} \right\rangle$$

where the *generator* σ_i exchanges the endpoints of strings i and $i+1$ with string i crossing over string $i+1$. In particular, all Lorenz braids can be expressed as products of these generators.

Each Lorenz braid β is a simple braid, so it has an associated permutation π. This permutation has only one cycle *iff* it is associated to a knot, and has k cycles if it is associated to a link with k components (knots).

Symbolic dynamics for the Lorenz map

Let $f^j = f \circ f^{j-1}$ be the jth iterate of the Lorenz map f and f^0 be the identity map. We define the *itinerary* of a point x under f as the symbolic sequence $(i_f(x))_j, j = 0, 1, \ldots$ where

$$(i_f(x))_j = \begin{cases} L & \text{if } f^j(x) < 0 \\ 0 & \text{if } f^j(x) = 0 \\ R & \text{if } f^j(x) > 0. \end{cases}$$

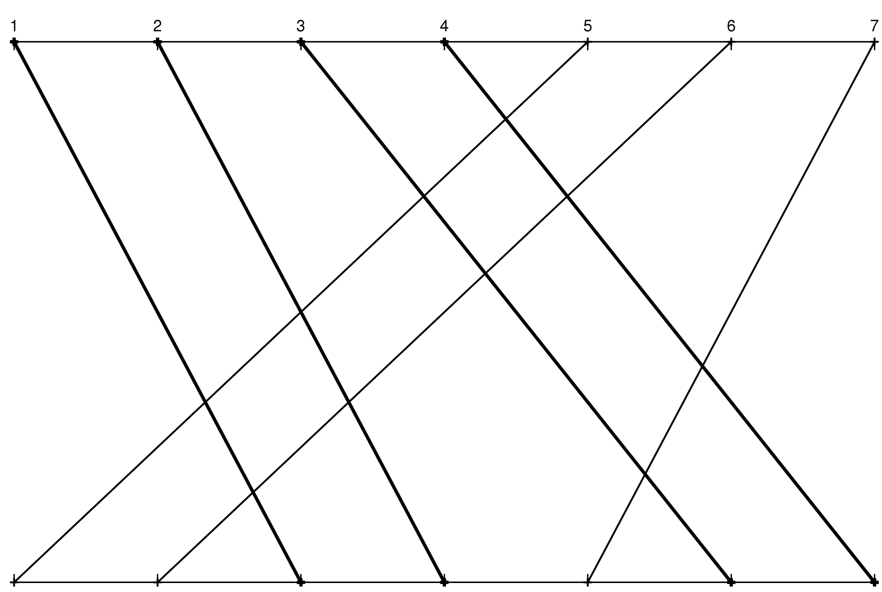

Fig. 3. A Lorenz braid.