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Numerical study of the generalised Klein–Gordon equations

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HIGHLIGHTS

- Derivation of generalised Klein–Gordon equations is presented in the context of deep water waves.
- Accuracy of this approximate model is studied using analytical and numerical methods.
- For travelling periodic waves the model is shown to be more accurate than the cubic Zakharov equations.
- Dynamics of periodic and localised wave trains is studied numerically.
- It is shown numerically that this model can develop Riemann-type wave breaking phenomenon.

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ABSTRACT

In this study, we discuss an approximate set of equations describing water wave propagating in deep water. These generalised Klein–Gordon (gKG) equations possess a variational formulation, as well as a canonical Hamiltonian and multi-symplectic structures. Periodic travelling wave solutions are constructed numerically to high accuracy and compared to a seventh-order Stokes expansion of the full Euler equations. Then, we propose an efficient pseudo-spectral discretisation, which allows to assess the stability of travelling waves and localised wave packets.

1. Introduction

The water wave problem counts today more than 200 years of history (see A. Craik (2004), [1]). Despite some recent progress [2–5], the complete formulation remains a mathematical difficult problem and a stiff numerical one. Consequently, researchers have always been looking for specific physical regimes which would allow to simplify the governing equations [6,7]. There are two main regimes which attracted a particular attention from the research community: shallow and deep water approximations [8,9].

If λ is a characteristic wavelength and h is an average water depth, the shallow water approximation consists to assume that $h/\lambda \ll 1$ or in other words, the water depth is much smaller compared to the typical wavelength. This regime is relevant in coastal engineering problems [10–12]. In open ocean only tsunami and tidal waves are in this regime [13,14].

The deep water approximation is exactly the opposite case when $h/\lambda \gg 1$, *i.e.* the water depth is much bigger than the typical wavelength. In practice, some deep water effects (defocusing type of the NLS equation) can already manifest when $kh = 2\pi h/\lambda \gtrsim 1.36$. This regime is relevant for most wave evolution problems in open oceans [15]. In the present paper, we present a detailed derivation of what we call a "generalised Klein–Gordon (gKG)" equations using a variational principle [16]. To our knowledge, it is a novel model in deep water regime. By making comparisons with the full Euler equations, we show that these equations can, on some peculiar features, outperform the celebrated cubic Zakharov (cZ) equations [17,18]. Recently, a novel so-called *compact Dyachenko–Zakharov* equation was proposed [19] which describes the evolution of the complex wave envelope amplitude in deep

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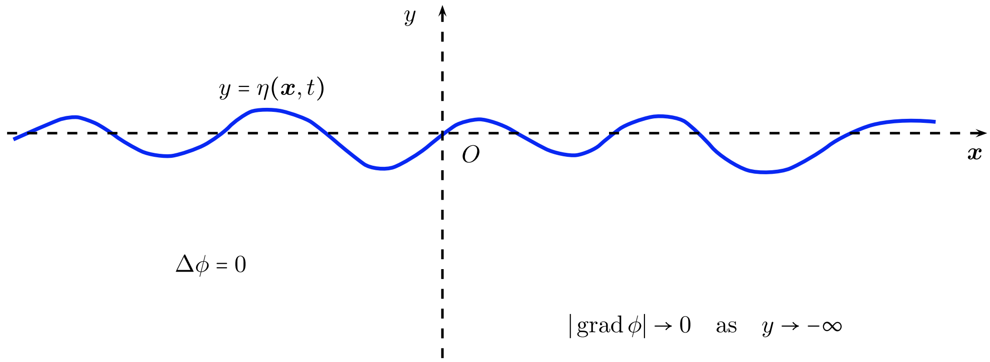

Fig. 1. Definition sketch of the fluid domain.

waters. This promising equation results from a sequence of thoroughly chosen canonical transformations, making the direct comparisons rather tricky.

The gKG equations have multiple variational structures. First of all, they appear as Euler–Lagrange equations of an approximate Lagrangian that possesses also a canonical Hamiltonian formulation [16]. In this study, we show that the gKG system can be recast into the multi-symplectic form [20,21] as well. The main idea behind this formulation is to treat the time and space variables on equal footing [22] while, for instance in Hamiltonian systems, the time variable is privileged with respect to the space. Based on this special structure, numerous multi-symplectic schemes have been proposed for multi-symplectic PDEs including the celebrated KdV and NLS equations [20,23–25]. These schemes are specifically designed to preserve exactly the discrete multi-symplectic form. However, these schemes turn out to be fully implicit, thence advantageous only for long time simulations using large time steps. Since in the present study we focus on the mid-range dynamics, we opt for a pseudo-spectral method which can insure a high accuracy with an explicit time discretisation [26,4,27,28]. Since the periodic and localised solutions play an important role in the nonlinear wave dynamics [29], we use the numerical method to study the behaviour of these solutions.

The present paper is organised as follows. In Section 2 we briefly present the essence of the deep water approximation and derive the gKG equations. In Section 3, we discuss some structural properties of the model and, in Section 4.1, we compare it to the classical cubic Zakharov (cZ) equations. Periodic travelling wave solutions are computed in Section 4. The numerical method for the gKG initial value problem is described in Section 5. Some numerical tests are presented in Section 6. Finally, the last Section 7 contains main conclusions of this study.

2. Mathematical modelling

Consider an ideal incompressible fluid of constant density ρ. The vertical projection of the fluid domain Ω is a subset of $\mathbb{R}^2$. The horizontal independent variables are denoted by $\boldsymbol{x} = (x_1, x_2)$ and the upward vertical one by y. The origin of the Cartesian coordinate system is chosen such that the surface $y = 0$ corresponds to the still water level. The fluid is bounded above by an impermeable free surface at $y = \eta(\boldsymbol{x}, t)$. We assume that the fluid is unbounded below. This assumption constitutes the so-called deep water limiting case which is valid if the typical wavelength is much smaller than the average water depth. The sketch of the physical domain is shown in Fig. 1.

2.1. Fundamental equations

Assuming that the flow is incompressible and irrotational, the governing equations of the classical water wave problem over an infinite depth are the following [30,9,6,7]:

$$\nabla^2\phi + \partial_y^2\,\phi = 0 \quad -\infty \leqslant y \leqslant \eta(\boldsymbol{x}, t), \tag{2.1}$$

$$\partial_t\eta + (\nabla\phi)\boldsymbol{\cdot}(\nabla\eta) - \partial_y\,\phi = 0 \quad y = \eta(\boldsymbol{x}, t), \tag{2.2}$$

$$\partial_t\phi + \frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,(\partial_y\phi)^2 + g\,\eta = 0 \quad y = \eta(\boldsymbol{x}, t), \tag{2.3}$$

$$|\mathrm{grad}\,\phi| \to 0 \quad y \to -\infty, \tag{2.4}$$

with ϕ being the velocity potential (*i.e.*, $\boldsymbol{u} = \nabla\phi$, $v = \partial_y\phi$), g the acceleration due to gravity and where $\nabla = (\partial_{x_1}, \partial_{x_2})$ denotes the gradient operator in horizontal plane.

The incompressibility condition leads to the Laplace equation for ϕ. The main difficulty of the water wave problem lies on the nonlinear free boundary conditions and that the free surface shape is unknown. Eq. (2.2) expresses the free-surface kinematic condition, while the dynamic condition (2.3) expresses the free surface isobarity. Finally, the last condition (2.4) means that the velocity field decays to zero as $y \to -\infty$.

The water wave problem possesses several variational structures [31–34]. In the present study, we will focus mainly on the Lagrangian variational formalism, but not exclusively. Surface gravity wave equations (2.1)–(2.3) can be derived as Euler–Lagrange equations of a functional proposed by Luke [31]

$$\mathcal{L} = \int_{t_1}^{t_2}\int_{\Omega} \mathscr{L}\,\rho\,\mathrm{d}^2\boldsymbol{x}\,\mathrm{d}t,$$

$$\mathscr{L} = -\int_{-\infty}^{\eta}\left[g\,y + \partial_t\,\phi + \frac{1}{2}\,(\nabla\,\phi)^2 + \frac{1}{2}\,(\partial_y\,\phi)^2\right]\mathrm{d}y. \tag{2.5}$$

In a recent study, Clamond and Dutykh [16] proposed to use Luke's Lagrangian (2.5) in the following relaxed form

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{\mu}}\boldsymbol{\cdot}\nabla\eta - \tilde{\nu})\,\tilde{\phi} - \frac{1}{2}\,g\,\eta^2 + \int_{-\infty}^{\eta}\left[\boldsymbol{\mu}\boldsymbol{\cdot}\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^2 + \nu\,v - \frac{1}{2}\,v^2 + (\nabla\boldsymbol{\cdot}\boldsymbol{\mu} + \nu_y)\,\phi\right]\mathrm{d}y, \tag{2.6}$$

where $\{\boldsymbol{u}, v, \boldsymbol{\mu}, \nu\}$ are the horizontal velocities, the vertical velocity and the associated Lagrange multipliers, respectively. The additional variables $\{\boldsymbol{\mu}, \nu\}$ (Lagrange multipliers) are called pseudo-velocities. The over 'tildes' denote a quantity computed at the free surface $y = \eta(\boldsymbol{x}, t)$.

While the original Lagrangian (2.5) incorporates only two variables (η and ϕ), the relaxed Lagrangian density (2.6) involves six variables $\{\eta, \phi, \boldsymbol{u}, v, \boldsymbol{\mu}, \nu\}$. These additional degrees of freedom provide us with more flexibility in constructing various approximations. For more details, explanations and examples we refer to [16].