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Synchronization of coupled chaotic maps

Georgi S. Medvedev*, Xuezhi Tang

Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, United States

HIGHLIGHTS

- We derive a sufficient condition for synchronization of coupled maps on graphs.
- The localization of the graph eigenvalues is shown to facilitate synchronization.
- The results are applied to coupled map networks on certain Cayley and random graphs.

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ABSTRACT

We prove a sufficient condition for synchronization for coupled one-dimensional maps and estimate the size of the window of parameters where synchronization takes place. It is shown that coupled systems on graphs with positive eigenvalues of the normalized graph Laplacian concentrated around 1 are more amenable for synchronization. In the light of this condition, we review spectral properties of Cayley, quasirandom, power-law graphs, and expanders and relate them to synchronization of the corresponding networks. The analysis of synchronization on these graphs is illustrated with numerical experiments. The results of this paper highlight the advantages of random connectivity for synchronization of coupled chaotic dynamical systems.

1. Introduction

Among problems concerning dynamics of large networks, synchronization holds a special place [1,2]. With applications ranging from the genesis of epilepsy [3] to stability of power grids [4,5], understanding principles underlying synchronization in real world networks is of the utmost importance. From the theoretical standpoint, the mathematical analysis of synchronization provides valuable insights into the role of the network topology in shaping collective dynamics.

The mechanism of synchronization in coupled dynamical systems in general depends on the type of dynamics generated by individual subsystems. For many diffusively coupled systems, the contribution of network organization to synchronization can be effectively described by the smallest positive eigenvalue of the graph Laplacian, also called the algebraic connectivity of the network [6]. This has been shown for coupled scalar differential equations (so-called, consensus protocols) [7,8], limit cycle oscillators [9], excitable systems forced by noise [10], and certain slow–fast systems [11]. Interestingly, stochastic stability of the synchronous state can be estimated in terms of the total effective resistance of the graph [12,13]. Synchronization of the coupled chaotic systems features a new effect. It turns out that the parameter domain for synchronization depends on both the smallest and the largest positive eigenvalues (EVs) of the graph Laplacian. This has been observed for systems of coupled differential equation models in [14,15] and for coupled map lattices [16]. Furthermore, the analysis in [16] shows that coupled systems on graphs, whose graph Laplacians have all positive EVs localized, are optimal for synchronization in the sense that they synchronize in a broader range of parameters compared to the systems on the graphs whose eigenvalues are spread out. The conclusions in [16] are based on formal linear stability analysis of synchronous solutions of coupled systems of chaotic maps.

The goal of the present paper is twofold. First, we rigorously prove a sufficient condition for synchronization for systems of coupled maps. Our condition is slightly weaker than that proposed in [16], but it admits a simple proof and captures the mechanism of chaotic synchronization. Specifically, it shows that diffusive

* Corresponding author.
E-mail addresses: medvedev@drexel.edu (G.S. Medvedev), xt32@drexel.edu (X. Tang).

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coupling counteracts intrinsic instability of chaotic system to produce stable spatially coherent solutions. The synchronizing effect of the coupling is more pronounced on graphs with localized positive eigenvalues. To this end, in the second part of this paper, we discuss spectral properties of certain symmetric and random graphs in the light of the sufficient condition for synchronization. Specifically, we review the relevant facts about the eigenvalues of Cayley graphs [17] and contrast their properties to those of quasirandom and power-law graphs [18,19] and expanders [20]. These results highlight the advantages of random connectivity for chaotic synchronization.

2. Stability analysis

Let $\Gamma = (V, E)$ be an undirected graph on n nodes. The set of nodes is denoted by $V = [n] := \{1, 2, \dots, n\}$. The edge set E contains (unordered) pairs of adjacent nodes from V. Unless stated otherwise, all graphs in this paper are assumed to be simple meaning that they do not contain loops ($ii \notin E \ \forall i \in V$) and multiple edges. Denote the neighborhood of $i \in V$ by

$$N(i) = \{j \in V : \ ij \in E\}.$$

The cardinality of $N(i)$ is called the degree of node $i \in V$ and denoted by $d_i = |N(i)|$. If $d_i = d \ \forall i \in [n]$ then Γ is called a d-regular graph.

At each node of Γ we place a dynamical system

$$x_{k+1} = f(x_k), \tag{2.1}$$

where f is a continuous function from the unit interval $I := [0, 1]$ to itself. Local dynamical systems at adjacent nodes interact with each other via diffusive coupling. Thus, we have the following coupled system

$$x^{(i)}_{k+1} = f(x^{(i)}_k) + \frac{\epsilon}{d_i} \sum_{j \in N(i)} \left(f(x^{(j)}_k) - f(x^{(i)}_k) \right), \quad i \in [n], k = 0, 1, 2, \dots, \tag{2.2}$$

where $\epsilon \geq 0$ controls the coupling strength.

If Γ is a lattice, (2.2) is called a coupled map lattice. In this paper, Γ can be an arbitrary undirected connected graph. For convenience, we rewrite (2.2) in the vector form

$$\mathbf{x}_{k+1} = K_\epsilon \mathbf{f}(\mathbf{x}_k), \quad K_\epsilon = I_n - \epsilon L, \tag{2.3}$$

where $\mathbf{x} = (x^{(1)}, x^{(2)}, \dots, x^{(n)})$ and $\mathbf{f}(\mathbf{x}_k) = (f(x^{(1)}), f(x^{(2)}), \dots, f(x^{(n)}))$. L stands for the normalized graph Laplacian of Γ:

$$L = I_n - D^{-1}A, \tag{2.4}$$

where I_n is the $n \times n$ identity matrix, $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$ is the degree matrix, and A is the adjacency matrix of Γ:

$$(A)_{ij} = \begin{cases} 1, & ij \in E, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

Thus,

$$(L)_{ij} = \begin{cases} 1, & i = j, \\ -d_i^{-1}, & ij \in E, \\ 0, & \text{otherwise}. \end{cases}$$

In general, L is not a symmetric matrix. However, it is similar to a symmetric matrix

$$\tilde{L} = D^{1/2} L D^{-1/2} = I_n - D^{-1/2} A D^{-1/2}.$$

Thus, the EVs of L are real and nonnegative by Gershgorin's Theorem [21]:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n. \tag{2.6}$$

Furthermore, the row sums of L are equal to 0. Thus, $\lambda_1 = 0$ and

$$\mathcal{D} = \mathrm{span}\{\mathbf{1}_n\}, \quad \text{where } \mathbf{1}_n := (1, 1, \dots, 1)^\intercal \in \mathbb{R}^n, \tag{2.7}$$

is the corresponding eigensubspace. If Γ is connected then 0 is a simple EV of L [6]. We will need the following properties of pseudosimilar transformations (cf. [12]).

Lemma 2.1. *Let L be the $n \times n$ Laplacian matrix of a connected graph Γ, whose EVs are listed in (2.6). Suppose $S \in \mathbb{R}^{(n-1)\times n}$ is such that $\ker(S) = \mathcal{D}$ and let S^+ denote the Moore–Penrose pseudoinverse of S [21].*

Then $\hat{L} = SLS^+$ is a unique solution of the matrix equation

$$\hat{L} S = SL. \tag{2.8}$$

The EVs of $\hat{L}$ counting multiplicities are

$$\lambda_2, \lambda_3, \dots, \lambda_n. \tag{2.9}$$

The eigenspaces of L corresponding to λ_i, $i = 2, 3, \dots, n$, are mapped isomorphically to the corresponding eigenspaces of $\hat{L}$ by S.

Proof. The statement of the lemma follows from Lemma 2.4 of [12].

In the remainder of this paper, we use

$$S = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}. \tag{2.10}$$

Note that S has full row rank and $\ker(S) = \mathcal{D}$. For a given $\mathbf{x} \in \mathbb{R}^n$, one can use $|S\mathbf{x}|$ to estimate the distance from $\mathbf{x}$ to $\mathcal{D}$. Specifically, for the projection of $\mathbf{x}$ to $\mathcal{D}$, $P_\mathcal{D}$, we have

$$|P_\mathcal{D} \mathbf{x}| = |S^+ S \mathbf{x}| \leq \|S^+\| \, |S\mathbf{x}|. \tag{2.11}$$

Here and below, $\|\cdot\|$ stands for the operator norm [21].

$\mathcal{D}$ is an invariant subspace of (2.3). Trajectories from this subspace correspond to the space homogeneous solutions of (2.3). Thus, we refer to $\mathcal{D}$ as a synchronous subspace. We are interested in finding conditions on ϵ, which guarantee (asymptotic) stability of $\mathcal{D}$. In the analysis below, we follow the approach developed for studying synchronization in systems with continuous time [9,12,10].

Theorem 2.2. *Let $f : I \to I$ be a twice continuously differentiable function and $\Gamma = (V, E)$ be a connected graph on n nodes. Suppose*

$$F = \max_{x \in I} |f(x)| > 1, \tag{2.12}$$

and the EVs of the graph Laplacian of Γ satisfy

$$\frac{\lambda_n}{\lambda_2} < \frac{1 + F^{-1}}{1 - F^{-1}}, \tag{2.13}$$

where λ_2 and λ_n denote the smallest and largest positive EVs of the normalized graph Laplacian L (cf. (2.6)), respectively.

Let $\mathbf{x}_k$, $k = 0, 1, 2, \dots$ denote a trajectory of (2.2) with

$$\epsilon \in \left(\lambda_2^{-1}(1 - F^{-1}), \ \lambda_n^{-1}(1 + F^{-1}) \right). \tag{2.14}$$

Then there exists $\delta > 0$ such that $|P_\mathcal{D} \mathbf{x}_k| \to 0$ as $k \to \infty$ provided $|P_\mathcal{D} \mathbf{x}_0| < \delta$.

For the proof of this theorem, we will need the following auxiliary lemma, which we state first.