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Hamiltonian formulation of the extended Green–Naghdi equations

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HIGHLIGHTS

- We derive the extended Green–Naghdi (GN) equations which incorporate the higher-order dispersion.
- We show that the extended GN equations have the same Hamiltonian structure as that of the GN equation.
- We prove that Zakharov's Hamiltonian equations of motion are equivalent to the extended GN equations.

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ABSTRACT

A novel method is developed for extending the Green–Naghdi (GN) shallow-water model equation to the general system which incorporates the arbitrary higher-order dispersive effects. As an illustrative example, we derive a model equation which is accurate to the fourth power of the shallowness parameter while preserving the full nonlinearity of the GN equation, and obtain its solitary wave solutions by means of a singular perturbation analysis. We show that the extended GN equations have the same Hamiltonian structure as that of the GN equation. We also demonstrate that Zakharov's Hamiltonian formulation of surface gravity waves is equivalent to that of the extended GN system by rewriting the former system in terms of the momentum density instead of the velocity potential at the free surface.

1. Introduction

The Green–Naghdi (GN) equation which is also known as the Serre or Su–Gardner equations models the fully nonlinear and weakly dispersive surface gravity waves on fluid of finite depth. See Serre [1], Su and Gardner [2] and Green and Naghdi [3]. Although the GN equation approximates the Euler equations for the irrotational flows, it exhibits several remarkable features. In particular, it has a Hamiltonian formulation which provides a unified framework in exploiting the mathematical structure of various model equations such as the Boussinesq, Korteweg–de Vries (KdV) and Camassa–Holm (CH) equations (Camassa and Holm [4] and Camassa et al. [5]). A large number of works have been devoted to the studies of the GN equation from both analytical and numerical points of view. A review article by Barthélemy [6] describes the derivation of the GN equation as well as a method for improving the dispersive effect. Furthermore, the improved model equations are tested against experiments. The recent article by Bonneton et al. [7] reviews the high-order numerical methods for the GN equation and the numerical results in comparison with breaking random wave propagation experiments. The following two monographs are concerned with the derivation and mathematical properties of the GN and other water wave equations: Constantin [8] provides an overview of some main results and recent developments in nonlinear water waves including breaking waves and tsunamis. Lannes [9] addresses the derivation of various asymptotic model equations and their mathematical analysis which is mainly devoted to the well-posedness of the model equations.

The GN equation incorporates the dispersion of order δ^2, where $\delta = h_0/l$ is the shallowness parameter (h_0: mean depth of the fluid, l: typical length scale of the wave). To improve the dispersion characteristics, various attempts have been made to extend the GN equation. Among them, the model equations have been derived which include the dispersive terms of order δ^4 (Kirby [10], Madsen and Schäffer [11,12] and Gobbi et al. [13]). Numerical computations have been performed for these equations to examine the wave profiles and the amplitude–velocity relations as well as the effect of dispersion on the wave characteristics. Note, however that whether the proposed higher-order dispersive model equations permit the Hamiltonian formulations has not been discussed so far. In this paper, we extend the GN equation which is accurate to the dispersive terms of order δ^{2n} while preserving the full nonlinearity, where n is an arbitrary positive integer. The case $n = 1$ corresponds to the GN equation. We show that the extended model equations have the same Hamiltonian structure as that of the GN equation.

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We consider the two-dimensional irrotational flow of an incompressible and inviscid fluid of uniform depth. The effect of surface tension is neglected for the sake of simplicity. The governing equation of the water wave problem is given in terms of the dimensionless variables by

$$\delta^2\phi_{xx} + \phi_{yy} = 0, \quad -\infty < x < \infty, \ -1 < y < \epsilon\eta, \tag{1.1}$$

$$\eta_t + \epsilon\phi_x\eta_x = \frac{1}{\delta^2}\phi_y, \quad y = \epsilon\eta, \tag{1.2}$$

$$\phi_t + \frac{\epsilon}{2\delta^2}(\delta^2\phi_x^2 + \phi_y^2) + \eta = 0, \quad y = \epsilon\eta, \tag{1.3}$$

$$\phi_y = 0, \quad y = -1. \tag{1.4}$$

Here, $\phi = \phi(x, y, t)$ is the velocity potential, $\eta = \eta(x, t)$ is the profile of the free surface, and the subscripts x, y, and t appended to ϕ and η denote partial differentiations. The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{x} = lx, \tilde{y} = h_0y, \tilde{t} = (l/c_0)t, \tilde{\eta} = a\eta$ and $\tilde{\phi} = (gla/c_0)\phi$, where a and c_0 are characteristic scales of the amplitude and velocity of the wave, respectively, and g is the acceleration due to the gravity. Note that $c_0 = \sqrt{gh_0}$ is the long wave phase velocity. In the problem under consideration, one can choose the two independent dimensionless parameters, $\epsilon = a/h_0$ and $\delta = h_0/l$. The former parameter characterizes the magnitude of nonlinearity whereas the latter characterizes the dispersion or shallowness.

In Section 2, we provide a recipe for deriving the model equations. See, for instance Matsuno [14] as for an analogous method which develops a procedure for obtaining the full dispersion model equations of the water wave problem. After completing the construction of the extended GN system, we derive as an example a model equation which is accurate to order δ^4. In Section 3, we show that the extended GN equations can be formulated in a Hamiltonian form by employing an appropriate Lie–Poisson bracket. At the same time, we demonstrate that the extended equations are equivalent to Zakharov's equations of motion for surface gravity waves. In Section 4, we briefly address the solitary wave solutions of the δ^4 GN model. Finally, Section 5 is devoted to conclusion.

2. Derivation of the extended Green–Naghdi equations

2.1. The extended GN system

We first introduce the mean horizontal velocity component $\bar{u} = \bar{u}(x, t)$ by

$$\bar{u}(x, t) = \frac{1}{h}\int_{-1}^{\epsilon\eta}\phi_x(x, y, t)dy, \quad h = 1 + \epsilon\eta, \tag{2.1}$$

where h is the total depth of the fluid. The horizontal and vertical components of the surface velocity u and v are given respectively by

$$u(x, t) = \phi_x(x, y, t)|_{y=\epsilon\eta}, \tag{2.2}$$

$$v(x, t) = \phi_y(x, y, t)|_{y=\epsilon\eta}. \tag{2.3}$$

Multiplying (2.1) by h and differentiating the resultant expression by x and then using (1.1), (1.4), (2.2) and (2.3), we obtain the relation $(h\bar{u})_x = \epsilon\eta_x u - v/\delta^2$, or since $\epsilon\eta_x = h_x$

$$v = \delta^2\{-(h\bar{u})_x + h_x u\}. \tag{2.4}$$

Substitution of (2.4) into (1.2) yields the evolution equation for h:

$$h_t + \epsilon(h\bar{u})_x = 0. \tag{2.5}$$

An advantage of choosing h and $\bar{u}$ as the dependent variables is that (2.5) becomes an exact equation without any approximation.

Next, we differentiate (2.2) and (2.3) by x and t to obtain the relations

$$u_x = \phi_{xx} + \epsilon\phi_{xy}\eta_x, \qquad u_t = \phi_{xt} + \epsilon\phi_{xy}\eta_t, \tag{2.6}$$

$$v_x = \phi_{xy} + \epsilon\phi_{yy}\eta_x, \qquad v_t = \phi_{yt} + \epsilon\phi_{yy}\eta_t, \tag{2.7}$$

where the derivatives $\phi_{xx}, \phi_{xy}, \phi_{yy}, \phi_{xt}$ and ϕ_{yt} are evaluated at $y = \epsilon\eta$. Similarly,

$$\left(\phi_t|_{y=\epsilon\eta}\right)_x = \phi_{xt} + \epsilon\phi_{yt}\eta_x. \tag{2.8}$$

Eliminating ϕ_{xt} and ϕ_{yt} with use of (2.6) and (2.7), (2.8) becomes

$$\left(\phi_t|_{y=\epsilon\eta}\right)_x = u_t + v_t h_x - v_x h_t. \tag{2.9}$$

If we differentiate (1.3) by x, insert (2.4) and (2.9) in the resultant expression and use (2.5), we obtain the evolution equation for u:

$$u_t + v_t h_x + \epsilon u u_x + \epsilon h_x u v_x + \eta_x = 0. \tag{2.10}$$

Using (2.5), Eq. (2.10) can be recast into the form

$$[h(u + vh_x)]_t + \epsilon[h(u + vh_x)\bar{u}]_x + \epsilon[hv(2h_x\bar{u}_x + h\bar{u}_{xx}) + h(u - \bar{u})(u_x + v_x h_x)] + h\eta_x = 0. \tag{2.11}$$

The system of Eqs. (2.5) and (2.10) (or (2.11)) is equivalent to the basic Euler system (1.1)–(1.4). To obtain the extended GN equations, one needs to express the variables u and v in (2.10) in terms of h and $\bar{u}$. This is always possible as will be exemplified below. Consequently, Eq. (2.10) can be written in the form $\bar{u}_t = \sum_{n=0}^{\infty}\delta^{2n}K_n$, where K_n are polynomials of h and $\bar{u}_{nx}, \bar{u}_{nx,t}$, $(\bar{u}_{nx} = \partial^n\bar{u}/\partial x^n, n = 0, 1, 2 \ldots)$. If one retains the terms up to order δ^{2n}, it yields the extended GN equation which is accurate to δ^{2n}. In accordance with this fact, we call the system of Eqs. (2.5) and (2.10) (or (2.11)) with h and $\bar{u}$ being the dependent variables the extended GN system.

2.2. The δ^4 model

Now, we derive the extended GN equation explicitly in the case of $n = 2$ by truncating the system of Eqs. (2.5) and (2.11) at order δ^4, which we call the δ^4 model. To this end, we express the solution of the Laplace equation (1.1) subjected to the boundary condition (1.4) in the form of an infinite series (see, for instance Whitham [15])

$$\phi(x, y, t) = \sum_{n=0}^{\infty}(-1)^n\delta^{2n}\frac{(y+1)^{2n}}{(2n)!}f_{2nx}, \quad f_{2nx} = \frac{\partial^{2n}f}{\partial x^{2n}}, \tag{2.12}$$

where $f = f(x, t)$ is the velocity potential at the fluid bottom $y = -1$. The expressions (2.1)–(2.3) then become

$$\bar{u} = \sum_{n=0}^{\infty}(-1)^n\delta^{2n}\frac{h^{2n}}{(2n+1)!}f_{(2n+1)x}, \tag{2.13}$$

$$u = \sum_{n=0}^{\infty}(-1)^n\delta^{2n}\frac{h^{2n}}{(2n)!}f_{(2n+1)x}, \tag{2.14}$$

$$v = \sum_{n=1}^{\infty}(-1)^n\delta^{2n}\frac{h^{2n-1}}{(2n-1)!}f_{2nx}. \tag{2.15}$$

Retaining the terms of order δ^4, (2.13) gives

$$\bar{u} = f_x - \frac{\delta^2}{6}h^2 f_{xxx} + \frac{\delta^4}{120}h^4 f_{xxxxx} + O(\delta^6). \tag{2.16}$$

The inverse relation in which f_x is expressed in terms of h and $\bar{u}$ is achieved by the successive approximation starting from $f_x = \bar{u}$. This leads to

$$f_x = \bar{u} + \frac{\delta^2}{6}h^2\bar{u}_{xx} + \delta^4\left\{\frac{h^2}{36}(h^2\bar{u}_{xx})_{xx} - \frac{h^4}{120}\bar{u}_{xxxx}\right\} + O(\delta^6). \tag{2.17}$$