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Exchange orbits in the planar 1 + 4 body problem

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HIGHLIGHTS

- We study doubly-symmetric orbits in the $1 + 2n$-body problem.
- We compute doubly-symmetric orbits of exchange type in the five-body case.
- The computed orbits belong to a 1-parameter family of time-reversible invariant tori.
- The initial conditions were determined by means of solving a boundary value problem.
- The 1-parameter family is related with a 4-gon solution.

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ABSTRACT

We study some doubly-symmetric orbits in the planar $1+2n$-body problem, that is the mass of the central body is significantly bigger than the other $2n$ equal masses. The necessary and sufficient conditions for periodicity of the orbits are discussed. We also study numerically these kinds of orbits for the case $n = 2$. The system under study corresponds to one conformed by a planet and four satellites of equal mass. We determine a 1-parameter family of time-reversible invariant tori, related with the reversing symmetries of the equations of motion. The initial conditions of the orbits were determined by means of solving a boundary value problem with one free parameter. The numerical solution of the boundary value problem was obtained using the software AUTO. For the numerical analysis we have used the value of 3.5×10^{-4} as mass ratio of some satellite and the planet. In the computed solutions the satellites are in mean motion resonance 1:1 and they librate around a relative equilibria, that is a solution where the distances between the bodies remain constant for all time.

1. Introduction

Among the different issues to study in the dynamics of celestial bodies, the search for periodic orbits is one of the most attractive. Certainly, some kind of nearly periodic motions require the presence of a big central body to ensure stability, making possible the corresponding motion. A well known example of this is the coorbital motion, that is two or more bodies in a 1:1 mean motion resonance. Some interesting orbits of coorbital type appear in the restricted three body problem, for instance the tadpole or horseshoe ones [1,2]. In the tadpole case the massless body orbits around the Lagrange points L_4 or L_5, as happens for the Trojan asteroids [3] in the Sun–Jupiter system. On the other hand, in the horseshoe orbit the test particle follows a path such that encompass the three Lagrange points L_3, L_4 and L_5. In the non-restricted case, that is the third body has non-negligible mass, the horseshoe shaped orbits also exist; the difference with respect to the restricted case is that the primary less massive has its own horseshoe shaped orbit (in a rotating frame), instead of having a fixed position in the rotating frame. An example of this phenomena occurs in the system conformed by Saturn, Janus and Epimetheus [4]. There exist another kind of coorbital orbits in the general planar three-body problem, closely related with the horseshoe ones, the so called exchange orbits [5]. These orbits are characterized by a periodic interchange of the semi-major axis (exchange-a), or eccentricities (exchange-e), of the orbiting bodies. The exchange-a orbits and the horseshoe ones in the general planar three body problem are equivalent.

In the following we give a description of the exchange-a orbits (cycle) in the general planar three-body problem (see Fig. 1); this description holds for the horseshoe orbits in the same problem. A

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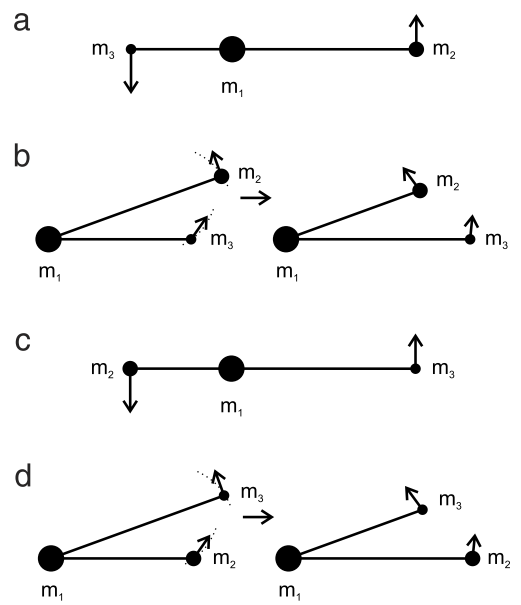

Fig. 1. Evolution of an exchange-a (horseshoe) orbit of the three-body problem in the inertial frame. (a) Initial alignment with the body 3 in an inner orbit. (b) Interchange of orbits during the *encounter* (3, 2). (c) Second alignment with the body 2 in an inner orbit. (d) The *encounter* (2, 3) after the alignment in (c).

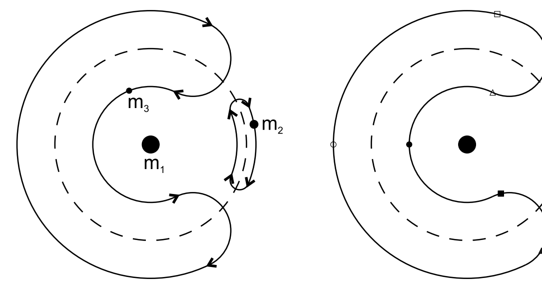

Fig. 2. Exchange-a (horseshoe) orbits (three-body problem) in a rotating frame with constant angular velocity, for the mass ratio $m_2/m_3 = 4$. At the right side are indicated the six configurations that make up Fig. 3. It was used a symbol per configuration. The correspondence is as follows: (a) bold circle, $(b)_1$ bold square, $(b)_2$ bold triangle, (c) blank circle, $(d)_1$ blank square and $(d)_2$ blank triangle.

detailed description of exchange-e orbits can be found in [6]. Consider a point P of the orbit where the configuration of the three bodies is collinear. Close to the point P, the interaction between the satellites is negligible and their motions are dominated by the force exerted by the primary body, then the trajectory of each satellite is approximately elliptic; we assume zero osculating eccentricities. At the starting point P we consider that the semimajor axis associated to the orbit of the particle 3 is smaller than the one corresponding to 2, therefore it is said that 3 is in an inner orbit, and 2 is in an outer one. The satellites rotate around the primary and eventually the distance between them decreases, and consequently the interaction between the minor bodies is increased which leads to a change in the orbits: 3 follows an outer orbit, and 2 an inner one, we say that a close *encounter* (3, 2) between the satellites has happened (we use the label (3, 2) in order to mark that in the corresponding *encounter* the third body passes from an inner orbit to an outer one, and vice versa for the second body). Notice that, since $r_{12} > r_{13}$ before the *encounter* (3, 2), and $r_{12} < r_{13}$ after the *encounter* (3, 2), the three bodies must pass through an isosceles configuration during the change, and that an alignment of the three bodies, in some way expected, does not occur. After the change of the orbits of the satellites, the separation between the minor bodies will increase until the three bodies reach an alignment where 2 is the inner body and 3 is the outer one, and the entire above process is repeated interchanging 2 and 3 in the discussion. This description corresponds to what we call a cycle since, qualitatively speaking, the orbit is conformed of cycles. In a rotating frame with an adequate constant angular velocity the trajectory of each satellite takes the form of a horseshoe, whose size depends on the minor masses. This property is outlined in Fig. 2 for the case $m_2/m_3 = 4$.

The tadpole and horseshoe orbits have been studied in the framework of the three-body problem, in particular the restricted model, with the aid of different mathematical tools (interested reader may refer to [7] and references therein). In [8–11] we have studied the horseshoe motion in the general planar three-body problem (we avoid the massless restriction on one of the bodies and the circular orbit prescribed for the other two, as it is assumed in the restricted circular planar three-body problem). In the periodic aspect, we have focused on the most symmetric orbits, those related with the reversing symmetries of the equations of motion. The symmetry leads to a simplification of the problem, in particular the reduction of the phase space. In this work we use a similar approach.

Many different aspects of coorbiting satellites have been studied, for instance planar central configurations and relative equilibria [12,13], or the dynamics of ring systems conformed by several satellites [14,15]. Nevertheless, the exchange character for N-body systems with $N > 3$ has not been contemplated. The main goal of this work is to establish the existence of exchange-a orbits for $N = 5$. To the best of our knowledge, this is the first time that this phenomenon is studied in the case $N = 5$. In fact, our main result given in Theorem 2 holds for $N = 1 + 4k$, with $k \in \mathbb{N}$, where we give a characterization of the most symmetric exchange-a orbits in the five-body problem. In particular, we establish the necessary and sufficient conditions for periodicity. In some cases the above orbits become subchoreographies, that is, solution where all the satellites describe the same trajectory preserving the corresponding time intervals. An interesting property of these orbits is that they are closely related to relative equilibria, and some special cases correspond exactly to relative equilibria.

This paper is organized as follows. In Section 2 we give the equations of motion and results concerning periodicity of certain doubly-symmetric orbits which appear in the $2n + 1$-body problem with $2n$ equal masses. We also introduce a special case of these kinds of orbits, namely subchoreographies. In Section 3 we describe the homographic solutions, the relative equilibria and the exchange-a orbits in the five-body problem, by analogy with the three-body case. Later, in Section 4 we establish the type of doubly-symmetric orbits to be studied, and use the symmetry of these orbits to simplify the problem. In Section 5 we explain the steps to follow in order to obtain numerically a doubly-symmetric exchange-a orbit in the five-body problem. This orbit will be used as "seed" of a boundary value problem with one free parameter which we solve numerically using the software AUTO [16]. The solution of the boundary value problem represents a 1-parameter family of time-reversible invariant tori, and each point on it defines a doubly-symmetric orbit which can be periodic or quasi-periodic in the inertial frame. In the same Section we exhibit an interesting property: the orbits within the family are in mean motion resonance 1:1, and librate around a relative equilibria solution of n-gon type. The numerical results are also given in that section. We give the conclusions of this work in Section 6.

2. Equations of motion, reversing symmetries and periodic motions

Consider N-point particles on the plane with positive masses m_i, denoted by short as $i = 1, \ldots, N$. The corresponding planar position and velocity vectors are given by

$$\mathbf{r}_i = (x_i, y_i), \qquad \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt}, \quad i = 1, \ldots, N,$$