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Controlling synchrony in a network of Kuramoto oscillators with time-varying coupling

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HIGHLIGHTS

- We use optimal control theory to investigate synchronization of Kuramoto oscillators.
- We assume synchrony is desirable, connectivity is costly, and coupling is dynamic.
- We derive and analyze necessary conditions for optimal networks.
- We show that dynamic coupling can be *more* efficient than static coupling.
- Optimal networks exhibit large amplitude oscillations and repulsion.

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ABSTRACT

The Kuramoto model describes the synchronization of a heterogeneous population of oscillators through a stationary homogeneous network in which oscillators are coupled via their phase differences. Recently, there has been interest in studying synchronization on time-varying networks, and time-varying generalizations of the Kuramoto network, in particular. Previous results indicate that networks with fast dynamics may be as efficient as static networks at promoting synchrony. In this paper we use optimal control theory to study synchronization on a time-varying Kuramoto network. Our results indicate that time-varying networks can be *more* efficient than static networks at promoting synchrony and show that fast network dynamics are not necessary for efficiency. In particular, we show that, near the synchronization threshold, time-varying networks can promote synchrony through slow oscillations that lengthen the duration of high synchrony states and shorten the duration of low synchrony states. Interestingly, repulsion is an essential feature of these optimal dynamic networks.

1. Introduction

Individuals within a population may synchronize their dynamics if they are able to interact via a network. The structure of the network and the form of the interaction that takes place on the network are important determinants of synchrony. Several authors have considered the relation between network structure and synchrony, under the assumption that the network is static [1–3]. More recently, there has been interest in studying synchrony on networks that change through time. Several papers have examined synchrony on networks in which the weights of the links/edges are dynamic [4–7]. Such networks can be used to model intermittent communication, as is the case for blinking networks in which the edges switch on and off according to some probability distribution [4], or more general networks that can switch between multiple states [7]. In [7], it was shown that, if the switching is fast enough, a population interacting over a switching network can achieve the same level of synchrony as a population interacting over a static network with the same time-averaged connectivity. Similarly, the blinking network considered in [4] was shown to support synchrony if the blinking is fast enough, and the threshold for synchronization was related to that of a corresponding averaged network. In addition, under certain conditions, it was shown that a blinking system may converge asymptotically to an attractor of the corresponding time-averaged system [5]. More recently still, there has been interest in evolving networks, i.e. in networks that change in response to the state of their nodes (see [8] for a recent

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review). Networks that adapt in order to promote synchronization are of particular interest [7,9,10]. In these works, the network dynamics are determined by a specific rule (e.g. edges are turned on and off with a given probability, the weight of an edge grows as the disparity between its vertices increases), and the dynamics of the resulting system are examined.

In this paper, we use optimal control theory in order to construct optimal time-varying networks. In particular, our work differs from previous research in that the network dynamics are identified using the necessary conditions associated with an optimal control problem. These networks are optimal in that they minimize/maximize objective functionals that depend on the synchrony of the population and the connectivity of the network, under the assumption that synchrony is beneficial and connectivity is costly.

In [3] we used optimal control theory to examine the properties of optimal static networks and, in particular, to determine how the heterogeneity of an optimal static network depends on the underlying heterogeneity of the population that the network joins. Due to the complexity of the optimal time-varying networks, this work focuses on characterizing optimal time-varying couplings between small groups of oscillators, beginning with a single oscillator pair. In the future, insights into the principles that govern optimal time-varying couplings between a small group of oscillators can be used to interpret results for large populations and identify size-dependent features of optimal time-dependent couplings. Here, we show that close to the synchronization threshold, dynamic networks are more efficient than static networks, be they heterogeneous or homogeneous, at promoting synchrony. In addition, numerical simulations indicate that dynamic networks use repulsion to enhance network synchrony, in one instance, even driving a contrarian oscillator to have negative velocity (See Fig. 6(b)), and lead to large amplitude oscillations of the network's order parameter.

Our analysis is based on the Kuramoto model, which is an elegant mathematical model of nonlinear phase oscillators that is capable of producing synchrony. The Kuramoto model is a system of nonlinear, coupled, ordinary differential equations [11]:

$$\frac{dx_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(x_j(t) - x_i(t)),$$

where N is the number of oscillators in the network, x_i is the phase of the ith oscillator, and ω_i is the natural velocity of the ith oscillator. The coefficient K is called the coupling strength. The complex valued order function,

$$r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{ix_j(t)},$$

quantifies the network's state. Here ψ denotes the centroid of the oscillator phases, and $r(t)$ provides a quantitative measure of network synchrony. In particular, $r(t) = 1$ corresponds to perfect synchrony, and $r(t) = 0$ corresponds to a state in which the oscillators' phases are evenly distributed around the unit circle [11]. We will refer to $r(t)$ as the order parameter of the system.

The state dynamics of the Kuramoto model can be rewritten in terms of the order parameter as follows:

$$\frac{dx_i}{dt} = \omega_i + r(t)K\ \sin(\psi(t) - x_i(t)),$$

so that each oscillator is coupled to the mean phase with strength $Kr(t)$ [11,12]. The network's dynamics depend on K as follows [12]. If K is small, the order parameter of the Kuramoto network fluctuates. There is a critical coupling threshold above which the oscillators become phase-locked and the order parameter stabilizes to a constant. As the coupling strength increases further, the value of the order parameter grows too. A more detailed account of the Kuramoto model and some bounds on the value of the critical coupling strength can be found in [12].

In the Kuramoto network model, the coupling strength is constant and is independent of time. In this paper, we consider a generalization of the Kuramoto model in which the coupling strength may vary between oscillator pairs and also through time. In particular, we consider the following Kuramoto type model with time varying-coupling:

$$\frac{dx_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} u_{ij}(t)\sin(x_j(t) - x_i(t)).$$

Assuming that network synchrony is desirable and that there is a cost associated with network connectivity, we use optimal control theory to design networks that can efficiently promote synchrony among a set of oscillators. The set of oscillators will be characterized by two vectors, x_0 and ω, that represent the oscillators' initial phases and natural velocities, respectively. Given x_0 and ω, we will use optimal control theory to find the coupling strengths, u_{ij}, that maximize an appropriate objective functional. In particular, the coupling strengths will serve as control variables, and their magnitudes as a measure of network connectivity. Note that although, in principle, the matrix of coupling strengths could have nonzero diagonal elements, the form of the state dynamics implies that the oscillators cannot influence their own dynamics. Hence, in an optimal network, for $i = 1, \ldots, N$, $u^*_{ii} = 0$. For this reason, when computing the cost of the Kuramoto network, we will also treat the diagonal elements as zero. For the purpose of formulating an objective functional, the network synchrony and the network connectivity cost are quantified as follows. The network synchrony is defined to be

$$\int_0^T r^2(t)dt,$$

and the connectivity cost over the interval $[0, T]$ is defined to be

$$\int_0^T \sum_{ij} u_{ij}(t)^2 dt.$$

The quadratic form of the coupling cost is important for identifying the optimal control and interpreting results. In particular, although intuitively the *total* connectivity of the network should determine the level of population synchrony, it is easily seen that for fixed total connectivity, i.e. for $\int_0^T \sum_{ij} \left|u_{ij}(t)\right| dt$ fixed, there can be multiple networks that achieve the same level of synchrony. The quadratic form of the connectivity cost, however, implies that among networks with the same total connectivity, those in which the connectivity is distributed evenly through time and among network edges have lower connectivity cost; i.e. the quadratic form of the connectivity cost penalizes dynamic and heterogeneous networks. Hence, temporal dynamics and network heterogeneity that are present in an optimal network must function to enhance the population's synchrony.

When comparing solutions, it is also useful to consider the average network synchrony and average connectivity cost:

$$\frac{1}{T}\int_0^T r^2(t)dt,$$

$$\frac{1}{TN(N-1)}\int_0^T \sum_{ij} u_{ij}(t)^2 dt.$$

Note that since the oscillators in a Kuramoto-type network cannot influence their own dynamics we have normalized the connectivity cost by $\frac{1}{N(N-1)}$, to reflect the fact that the system has, in effect, only $N(N-1)$ controls. For the same reason, when computing the