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The nonlinear interaction of convection modes in a box of a saturated porous medium

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HIGHLIGHTS

- We model convection in a finite box of porous media where three modes are viable.
- Each box is classified into one of two classes, with one exception.
- The bifurcation behaviour for an example from each class is studied.
- The behaviour for each box is inferred from these examples.
- The results qualitatively agree with examples in the literature.

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ABSTRACT

A plethora of convection modes may occur within a confined box of porous medium when the associated dimensionless Rayleigh number R is above some critical value dependent on the geometry. In many cases the crucial Rayleigh number R_c for onset is different for each mode, and in practice the mode with the lowest associated R_c is likely to be the dominant one. For particular sizes of box, however, it is possible for multiple modes (typically three) to share a common R_c. For box shapes close to these special geometries the modes interact and compete nonlinearly near the onset of convection. Here this mechanism is explored and it is shown that generically the dynamics of the competition takes on one of two possible structures. A specific example of each is described, while the general properties of the system enables us to compare our results with some previous calculations for particular box dimensions.

1. Introduction

Natural convection can occur in a porous medium if the buoyancy forces within the fluid are sufficiently strong to overcome viscous drag. Convection assists in efficiently transporting heat and is of importance in many applications including the geothermal energy industry. By targeting hot upwellings of fluid for extraction, the bore depth and the cost of a project can be reduced. One motivation for the present study is to gain further understanding of the convection patterns that may occur in finite domains thereby assisting in the location of geothermal upwellings.

Convection in porous media was first studied by Horton and Rogers [1] and Lapwood [2] who considered a layer of saturated porous media of unbounded horizontal extent. Linear stability analysis shows that if the value of the dimensionless Rayleigh number R (which is defined formally in (6)) exceeds the critical value of $4\pi^2$, then convection begins. Furthermore, the preferred pattern is one in which the convection cells span the full height of the layer and have a square-like profile. Later work by Beck [3] extended this model by restricting the domain to a three-dimensional finite box with no-flux conditions imposed on the sides. Then in order to satisfy the boundary conditions the horizontal mode-numbers of the possible solutions are no longer unrestricted but instead belong to a certain discrete spectrum of values. If we let p and q denote the mode numbers in the directions parallel to the horizontal edges of the box, then the critical Rayleigh number R_c depends on p and q and on the respective dimensionless lengths of the box L_x and L_y. In particular

$$R_c(p,q) = \frac{\pi^2\left(1+l^2\right)^2}{l^2} \quad \text{where } l^2 \equiv \left(\frac{p}{L_x}\right)^2 + \left(\frac{q}{L_y}\right)^2. \tag{1}$$

The solution mode with the lowest critical Rayleigh number is identified as the "preferred mode" and is clearly dependent on

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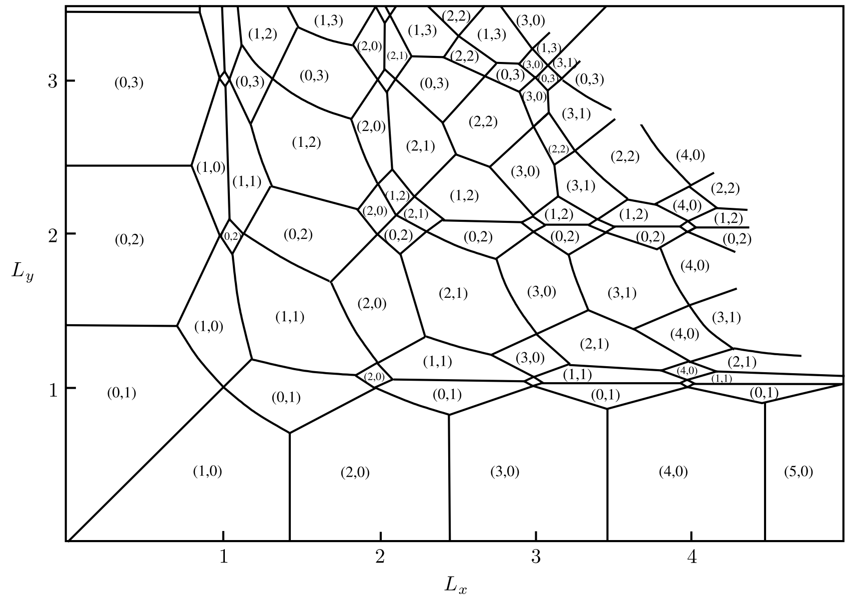

Fig. 1. (After Beck [3].) The preferred modes at the onset of convection, as a function of the horizontal aspect ratios of the box L_x and L_y. The identity of the preferred mode is denoted by the pair of horizontal mode-numbers (p, q).

L_x and L_y. Beck [3] calculated the form of R_c and his somewhat intricate result is reproduced in Fig. 1. This shows that the identity of the preferred mode is not a simple matter and the L_x/L_y parameter space is divided into many regions, some of which are quite small in extent. There is obvious symmetry about the line $L_x = L_y$ where the horizontal domain is square. It is not surprising that when the plan of the box is very far from square, with one of L_x or L_y much larger than the other, then the preferred mode is two-dimensional with convection rolls aligned perpendicular to the longer side. Apart from this behaviour, there is little else in the structure of Fig. 1 that is easily explained. The many points in L_x/L_y space where regions meet indicate box dimensions for which more than one mode share the same critical R_c. Near these crossings the interaction of modes can be described by a weakly nonlinear theory and an exploration of this phenomenon is the main aim for this work.

There have been many numerical studies of pattern selection in rectangular boxes including those by Horne and his colleagues [4–6], Straus and Schubert [7,8] and Riley and Winters [9]. Some corresponding analytical studies may be found in [10,11] and [12]. Particular mention should be made of the results of Riley and Winters [13] who studied the bifurcation process of interacting modes in a two-dimensional box as the Rayleigh number increases. Later work [14] introduced the refinement of sidewall heat imperfections, which breaks the symmetry of the pitchfork bifurcations. Borkowska–Pawlak and Kordylewski [15,16] explored the effect of changing the Prandtl values in a square box and in [17] used a Galerkin approach to uncover the bifurcations that may occur in a three-dimensional box. Zebib and Kassoy [18] calculated heat transfer rates of various modes and thereby concluded that two-dimensional "roll" patterns are preferred over three-dimensional "cell" patterns, at least at relatively small Rayleigh numbers. While our focus here is on describing the behaviour of convection in a box as the aspect ratios are allowed to vary, it is acknowledged that other physical effects also influence the pattern selection problem. Vincourt [19] studied the effect of non-uniform heating on a system while the introduction of heterogeneous porosity was the subject of [20].

Steen [21] used an eigenfunction expansion technique to determine the eventual steady-state solution of convection in a box with $L_x = L_y = 2^{1/4}$. This choice, while initially seeming somewhat strange, was motivated by some of the results in Fig. 1. It is noted that a box with these particular aspect ratios is one of critical dimensions in as much that the $(1, 0)$, $(0, 1)$ and $(1, 1)$ modes all share a common onset Rayleigh number. By reducing the problem to a system of ordinary differential equations it was demonstrated that there are a number of finite-amplitude stable solutions; which is eventually realised depends on the initial state of the system. Other solutions were also detected, including a conduction solution and configurations in which multiple modes co-exist but all of these are unstable. Steen's method of analysis is underpinned by a neat transformation whereby the nullcline surfaces become linear planes. Such a transformation is not implemented in this paper as it assumes that no symmetry-breaking effects occur, which is not necessarily the case here. Other examples are also considered, including a cubic box ($L_x = L_y = 1$) and a slightly stretched box.

Florio [22] elucidated the qualitative evolution of convection modes in boxes for which three modes share identical critical Rayleigh numbers. He used perturbation methods to derive coupled equations for the amplitudes A_i ($i = 1, 2, 3$) of the respective modes; these equations take the generic form

$$\frac{dA_i}{d\tau} = A_i\left(\alpha_i - \sum_{j=1}^{3}\gamma_{ij}A_j^2\right) \quad \text{for } i = 1, 2, 3 \tag{2}$$

where τ is a dimensionless time variable and the coefficients α_i and γ_{ij} are all positive and which depend on the particular modes under investigation. These equations admit stable solutions in which precisely one of the modes is non-zero along with a conduction solution in which each $A_i = 0$ and other solutions for which two of the modes do not vanish; however these latter solutions are unstable.

Considering the amplitude equations (2), it can be shown that in addition to the three primary modes there must exist three secondary solutions for which precisely two modes are simultaneously present [13,23]. Thus we conclude that there are three primary and three secondary solutions for every example. There is, however, one aspect which may differ between examples; depending on the values of the coefficients in the system (2) it is sometimes possible to also have a tertiary fixed point solution for which all three modes do not disappear. It transpires that the dynamics of the system can be distinguished depending on whether or not