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# Elliptical optical solitary waves in a finite nematic liquid crystal cell

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## HIGHLIGHTS

- Elliptically shaped solitary waves without orbital angular momentum (OAM) are unstable.
- We study the effects of OAM on elliptically shaped solitary waves.
- A shelf of radiation forms under the solitary wave, and radiates momentum.
- Modulation theory with mass and momentum losses compares well with full numerical solutions.
- "Chirp" solution cannot incorporate losses, and hence only trajectory compares well.

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## ABSTRACT

The addition of orbital angular momentum has been previously shown to stabilise beams of elliptic cross-section. In this article the evolution of such elliptical beams is explored through the use of an approximate methodology based on modulation theory. An approximate method is used as the equations that govern the optical system have no known exact solitary wave solution. This study brings to light two distinct phases in the evolution of a beam carrying orbital angular momentum. The two phases are determined by the shedding of radiation in the form of mass loss and angular momentum loss. The first phase is dominated by the shedding of angular momentum loss through spiral waves. The second phase is dominated by diffractive radiation loss which drives the elliptical solitary wave to a steady state. In addition to modulation theory, the "chirp" variational method is also used to study this evolution. Due to the significant role radiation loss plays in the evolution of an elliptical solitary wave, an attempt is made to couple radiation loss to the chirp variational method. This attempt furthers understanding as to why radiation loss cannot be coupled to the chirp method. The basic reason for this is that there is no consistent manner to match the chirp trial function to the generated radiating waves which is uniformly valid in time. Finally, full numerical solutions of the governing equations are compared with solutions obtained using the various variational approximations, with the best agreement achieved with modulation theory due to its ability to include both mass and angular momentum losses to shed diffractive radiation.



## 1. Introduction

The propagation of a bulk optical solitary wave in a nematic liquid crystal, a so-called nematicon [1], has become an active area of study [2,3] since their first experimental demonstration [4]. These studies are more general, however, as the equations governing bulk optical solitary waves in a nematic liquid crystal also apply to bulk solitary waves in thermal media [5], photorefractive crystals and other optically active bulk media [6]. A similar system of equations to that governing these bulk optical solitary waves arises in $\alpha$ models of fluid turbulence [7,8]. Nearly all of these optical studies have dealt with circularly symmetric beams, however. Elliptical bulk optical solitary waves introduce new mechanisms and effects not encountered with circularly symmetric beams.

The propagation of an elliptical cross-section beam in local media has been an experimental [9,10] and theoretical issue [11–13]. In local media, such beams are described by nonlinear

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Schrödinger (NLS)-type equations [11–13]. In addition to the standard instability of two dimensional solitary waves governed by nonlinear Schrödinger equations [6,14], there is an additional instability of elliptical beams due to the existence of the different major and minor axes of such optical beams, as the amount of nonlinearity required to support a radially symmetric solitary wave is dependent on the peak beam intensity [10]. Hence, the peak beam intensity also determines the diffraction angle that must balance with the self-focusing of the optical beam to self-trap [1,2] and thus form a solitary wave. For an optical beam to self-trap, radial symmetry is then required. However, an elliptical beam is asymmetric and, hence, difficulties arise in the support required for the two competing diffraction angles [10,11]. The term elliptic solitary wave will be used from here on to describe an elliptical cross-section solitary wave. Further, adding to the difficulty in forming an elliptic solitary wave, it has been shown both experimentally [9,10,15] and theoretically [11–13] that the widths of the elliptic beam periodically oscillate, as would be expected from the general behaviour of beams for NLS-type equations.

Several methods have been suggested to aid in the formation and stabilisation of an elliptic solitary wave before it diffracts into a circularly symmetric beam. Examples are to use partially incoherent elliptic beams with an anisotropic mutual coherence function [9,10], a medium with a nonlocal response [12,16] or applying an orbital angular momentum to the elliptic-shaped beam [13]. The propagation of an elliptic solitary wave in the nonlocal medium of a nematic liquid crystal (NLC) [2] is the subject of this work.

Elliptic solitons have been shown to exist in a nematic liquid crystal [16], which is an example of a nonlocal, nonlinear medium [2]. The key to the behaviour of an elliptic solitary wave in a nematic liquid crystal is the self-focusing response of an NLC. A nematic molecule tends to align itself with the direction of an electric field, whether this is an external bias field applied across the liquid crystal cell or that of an optical beam input into the cell [2,17]. If the optical beam is of sufficient power to overcome the Freédericksz threshold [2,17–19] the nematic molecules will rotate, thus altering the refractive index of the NLC. If the refractive index increases, this self-focusing response of the beam can balance diffraction, resulting in a solitary wave, or nematicon [2,3]. In addition, it has been shown that nematic molecules tend to align with the major axis of an elliptic beam [16]. However, the issue of the stability of elliptic beams has not been addressed.

In the present work, an elliptical cross-section optical beam with orbital angular momentum propagating through a finite sized nonlocal NLC cell is studied. As stated above, to induce the self-focusing response of the NLC, the optical beam intensity must be above the minimum to enable the nematic molecules to rotate, the Freédericksz threshold [2,17–19]. To enable the use of milliwatt beam powers a pre-tilt is induced within the NLC so the molecules form an angle $\theta_0 \sim \pi/4$ with the optical wavefront, with the Freédericksz threshold reduced to exactly zero when $\pi/4$. In this manner, milliwatt optical beam powers induce a sufficient change in the nematic's refractive index [20] to enable a nematicon to form. There are two main techniques for applying the desired pre-tilt angle. The first is to apply an external static electric field perpendicular to the optical axis in the direction of polarisation of the optical field. The second technique creates a static charge on the cell walls by "rubbing" them, thus causing the nematic molecules near the cell walls to rotate [2]. This tilt angle is then transferred throughout the bulk of the NLC cell by the intermolecular elastic links [2]. Rubbing the cell walls to pre-tilt the nematic molecules results in different decay rates of the nematic response to the optical beam. In one transverse dimension a linear decay is experienced [21], while in two transverse dimensions [22,23] a logarithmic decay results. This implies that the nematic response to the beam extends to the boundaries of the NLC cell and, as a result, the inclusion of proper boundary conditions is vital in order to model an elliptic solitary wave accurately.

The present work will focus on the role diffractive radiation and orbital angular momentum shed to diffractive radiation play in the evolution of an elliptical nematicon. While it has been found that angular momentum can stabilise an elliptical nematicon for short evolution distances [13], typically ∼5–10 revolutions of the major axis, it has been found that on longer scales, which amount to ∼100 rotations, that angular momentum loss to diffractive radiation causes the elliptical solitary wave to become circular and stop rotating. In addition to this effect of shed diffractive radiation, the effect the boundaries have on the evolution of elliptic solitary waves will also be investigated. This analysis will be based on using an exact solution for the director distribution and modulation equations [24] for the optical field derived using suitable trial functions [25,26] in a Lagrangian representation of the governing equations. Modulation theory has proved to be a successful technique for modelling the evolution of nonlinear optical beams in NLC, giving excellent agreement with full numerical solutions of the governing equations [27–33] and with experimental results [34–36]. In the present work we show how the shed diffractive radiation can be studied using geometric optics. We obtain approximate evolution equations for the elliptical solitary wave parameters which explain the relevant features of the processes observed in numerical solutions.

## 2. Governing equations

Consider a polarised, coherent elliptical cross-sectional optical beam input into a finite sized NLC cell. Let us take the $z$ direction as the propagation direction. The nematic molecules are arranged in a planar configuration within the NLC cell. The optical beam is polarised in the $x$ direction, which results in molecular rotation in the $(x, z)$ plane [1,2,4,37,38]. The nematic molecules are pre-tilted by an angle $\theta_0 \sim \pi/4$ in the $(x, z)$ plane [20], enabling the use of milliwatt beam powers, as the Freédericksz threshold is thus overcome [2,17–19]. The pre-tilt of the nematic is achieved by rubbing the cell walls. The intermolecular elastic forces of the NLC pass the rotation then achieved through the bulk of the medium, thus obtaining a semi-uniform pre-tilt. The optical beam's electric field causes a further rotation of the director by an angle $\theta$, so that the total director angle is given by $\phi = \theta_0 + \theta$, relative to the $z$ axis. The perturbation of the director due to the optical beam is small for milliwatt beam powers, $|\theta| \ll |\theta_0|$. The non-dimensional equations governing the propagation of the optical beam in this small extra rotation limit in the paraxial approximation are a strongly coupled pair of partial differential equations (PDEs), the first of which is an NLS-like equation for the optical beams and the second is Poisson's equation for the director rotation [21,22,39,40], these being

$$iE_z + \frac{1}{2}\nabla^2 E + 2\theta E = 0, \tag{1}$$

$$\nu\nabla^2\theta + 2|E|^2 = 0. \tag{2}$$

The Laplacian $\nabla^2$ is in the $(x, y)$ plane and $E$ is the complex valued envelope of the electric field. The elastic response of the NLC is given by the nonlocality parameter $\nu$, which is experimentally $O(100)$ [34]. In experiments, the optical beam experiences a phenomenon known as walk-off due to the refractive index being a tensor [24], whereby the optical beam deviates from the input wavevector along the $z$ direction and follows the beam's Poynting vector. This walk-off has been removed from the electric field equation (1) by using a phase transformation of the electric field [27]. The NLC cell is finite sized and is a rectangle with dimensions $0 \le x \le L_x$ and $0 \le y \le L_y$.