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Modulational instabilities of periodic traveling waves in deep water

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HIGHLIGHTS

- The spectrum of traveling water waves is asymptotically approximated, considering two and three-dimensional perturbations.
- A multiple scale expansion is employed, coupling wave slope to Bloch parameter.
- The radius of the disc of analyticity of the spectrum is predicted, and compared to numerical simulations.
- Modulational instabilities are computed.

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ABSTRACT

The spectrum of periodic traveling waves in deep water is discussed. A multi-scale method is used, expanding the spectral data and the Bloch parameter in wave amplitude, to compute the size and location of modulated instabilities. The role of these instabilities in limiting the spectrum's analyticity is explained. Both two-dimensional and three-dimensional instabilities are calculated. The asymptotic predictions are compared to numerical simulations.

1. Introduction

The spectral stability of traveling deep-water waves is studied. The water wave stability problem has a rich history, with great strides made in the late sixties with the work of Benjamin and Feir [1] and the development of Resonant Interaction Theory (RIT) [2–5]. The predictions of RIT have since been leveraged heavily by numerical methods; the influential works of MacKay and Saffman [6] and McLean [7] led to a taxonomy of water wave instabilities based on RIT. The most recent review article is that of Dias and Kharif [8]; since the publication of this review a number of modern numerical stability studies have been conducted [9–13].

In RIT, wave dynamics are studied using approximate models for the evolution of the amplitude of a small number of weakly nonlinear wave modes: triads, quartets, etc. Examples of such model equations which include modulational effects are the nonlinear Schrödinger equation, the Dysthe equation, and the Davey–Stewartson/Benney–Roskes equations [4,14–16]. The stability of Stokes waves has been studied in such models, often they were derived for exactly this purpose [17–19].

In this work, the spectrum of traveling water waves is approximated via small amplitude asymptotic expansion. This approach contrasts the vast majority of spectral stability computations, which are primarily numerical. Typical numerical computations calculate the spectrum via an eigensolver at each fixed amplitude—see for example [9,20,10]. Boundary perturbation methods take an alternative approach, expanding the traveling wave and spectrum in amplitude. Boundary perturbation methods calculate the coefficients of a series representation of the spectrum. Boundary perturbation methods have been applied to compute water waves numerous times [21–24] and are reviewed in [25]. This approach is employed for the water wave spectrum in [26,27].

In a series of recent works, the author and collaborators derived the weakly nonlinear asymptotics of the spectrum in conjunction with the development of boundary perturbation methods, for deep water gravity waves in [13], including surface tension in [28], and with finite depth effects in [29]. Each article in this series considers a two-dimensional fluid, and expands the spectrum in amplitude at a sampling of fixed Bloch parameters; instabilities which have both fixed Bloch parameter and are analytic in amplitude are observed to be rare.

It is known that small amplitude instabilities bifurcate from a set of resonant configurations, whose locations may be predicted

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by linear theory [7,8]. These resonant configurations are the Bloch parameters for which the spectrum of the linearization about the flat-state contains eigenvalue collisions. It is known that the spectrum is analytic as a function of amplitude at all Bloch parameters where the flat-state eigenvalues are simple [30]. Recent numerical simulations suggest that the spectrum is also analytic at eigenvalue collisions, but that the radius of the disc of analyticity vanishes as the Bloch parameter approaches the resonant configurations [28,29]. This vanishing radius has been proposed as a mechanism for *detecting* instabilities [27].

The potential flow equations have traveling wave solutions which are analytic in wave slope [31]. The leading asymptotics of the traveling wave have been computed numerous times [32–35]. The asymptotics of the spectrum have been computed for a two-dimensional fluid in [13,28,29], these asymptotics all compute the spectrum with *fixed* Bloch parameter. In this work, instabilities are computed with Bloch parameters which depend on amplitude, on both a two-dimensional and three-dimensional fluid. We use a multi-scale expansion which couples frequency and amplitude in a manner analogous to the modulational ansatz which is typically used to derive envelope equations [14,36,16]; we refer to the unstable spectral data computed in this manner as modulational instabilities.

Often the term modulational is used to refer only to the Benjamin–Feir instability. Although much of the asymptotic work regarding Benjamin–Feir dates back to the 1960s and RIT, more recently a number of authors have been pursuing rigorous proof of the existence of this instability in a variety of wave models [37,38]. Most similar in spirit to this work is that in [39,40], where an analogous perturbation in Bloch parameter is used. Although we consider only formal asymptotics, such asymptotics have been used as the basis for proofs of the existence of solutions in the TFE framework [31,30].

In the current framework, the classic long wave modulational instability, Benjamin–Feir, is recovered as are many other modulational instabilities. The onset of instability at fixed Bloch parameter, and thus the amplitude at which the fixed Bloch parameter spectrum loses analyticity, is predicted. Both two-dimensional and three-dimensional perturbations are considered. The asymptotic instability locations are compared with the numerical estimates of these locations using the method of Akers and Nicholls [13,28].

The paper is organized as follows. Section 2 begins by introducing the spectral stability problem for water waves. The asymptotics of the spectrum are then presented, in subsections organized by the type of resonance which is responsible for the flat-state eigenvalue collisions. Triad resonance is presented in Section 2.1; quartets are discussed in Section 2.2. We comment on higher order resonances in 2.3, and finally compute the Benjamin–Feir instability in Section 2.4.

2. Modulational instabilities of deep water waves

The widely-accepted model for irrotational motions of a large body of deep water in the absence of viscosity is the Euler equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad z < \epsilon\eta, \tag{2.1a}$$

$$\phi_z = 0, \quad z \to -\infty, \tag{2.1b}$$

$$\eta_t + \epsilon(\eta_x\phi_x + \eta_y\phi_y) = \phi_z, \quad z = \epsilon\eta, \tag{2.1c}$$

$$\phi_t + \frac{\epsilon}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \eta - \sigma\left(\frac{\eta_{xx} + \eta_{yy}}{\left(1 + \epsilon^2(\eta_x^2 + \eta_y^2)\right)^{3/2}}\right) = 0, \quad z = \epsilon\eta, \tag{2.1d}$$

where η is the free-surface displacement and ϕ is the velocity potential. System (2.1) has been nondimensionalized as in [13,41]. We assume that the wave slope, $\epsilon = A/L$ is small (A is a typical displacement and L, the characteristic horizontal length, is chosen so that waves in (2.1) have spatial period 2π). The constant $\sigma = \frac{\gamma}{gL^2}$ is a Bond number comparing the relative importance of gravity, g, to surface tension γ.

The potential flow equations (2.1), have traveling wave solutions which depend analytically on wave slope [31]. These solutions can be written in terms of the speed c, the displacement η, and the free surface trace of the potential Φ, each as a series in ϵ. Periodic traveling wave solutions are often called Stokes' waves, as the leading order terms of this series were first written by Stokes [32]. We consider the stability of the classic Stokes wave, which is constant in transverse direction, and at leading order is supported at wavenumber $k_0 = (1, 0)$. The speed, displacement and free surface trace of the potential of this wave are, to $O(\epsilon^2)$,

$$c = \sum_{n=0}^{\infty}\epsilon^n c_n = \begin{pmatrix}\sqrt{1+\sigma}\\ 0\end{pmatrix} + \epsilon^2\begin{pmatrix}\dfrac{2\sigma^2+\sigma+8}{4(1-2\sigma)\sqrt{1+\sigma}}\\ 0\end{pmatrix} + O(\epsilon^3), \tag{2.2a}$$

$$\bar{\eta} = \sum_{n=1}^{\infty}\epsilon^n\bar{\eta}_n = \epsilon e^{ik_0\cdot\mathbf{x}} + \epsilon^2\left(\frac{1+\sigma}{1-2\sigma}\right)e^{2ik_0\cdot\mathbf{x}} + * + O(\epsilon^3), \tag{2.2b}$$

$$\bar{\Phi} = \sum_{n=1}^{\infty}\epsilon^n\bar{\Phi}_n = \epsilon i\sqrt{1+\sigma}e^{ik_0\cdot\mathbf{x}} + \epsilon^2\left(\frac{3i\sigma\sqrt{1+\sigma}}{1-2\sigma}\right)e^{2ik_0\cdot\mathbf{x}} + * + O(\epsilon^3). \tag{2.2c}$$

In (2.2), the $*$ refers to the complex conjugate of the preceding terms. This traveling wave solution is constant in the transverse horizontal direction, here y. Later the perturbations of this wave will be permitted to have non-trivial dependence on both horizontal coordinates, $\mathbf{x} = (x, y)$.

The spectral stability of these traveling waves (2.2) is considered by writing Eq. (2.1) in terms of the free surface trace Φ and displacement η, as in [42], then substituting the ansatz

$$\begin{aligned}\eta &= \bar{\eta}(\mathbf{x} + ct) + \delta\zeta(\mathbf{x} + ct)e^{\lambda t}, \quad \text{and}\\ \Phi &= \bar{\Phi}(\mathbf{x} + ct) + \delta u(\mathbf{x} + ct)e^{\lambda t},\end{aligned} \tag{2.3}$$

and neglecting quadratic powers of δ. The result is a generalized spectral problem of the form

$$A(\bar{\eta}, \bar{\Phi}, c)w = \lambda B(\bar{\eta}, \bar{\Phi}, c)w, \tag{2.4}$$

where $w = (\zeta, u)^T$. Traveling waves are considered spectrally unstable if solutions to (2.4) have λ with positive real part. It is straightforward to calculate the operators A and B, and we refer the interested reader to [13,28].

To solve (2.4), we must append boundary conditions for w. Rather than assuming perturbations w share a period with the traveling waves, thus restricting to superharmonic perturbations [43], we consider arbitrary periods, including subharmonic perturbations [44]. Subharmonic perturbations satisfy Bloch (quasi) periodic boundary conditions [45]. If the traveling wave is x-periodic with period L, then the perturbations satisfy

$$w(x + L, y) = e^{ipL}w(x, y). \tag{2.5}$$

For $L = 2\pi$-periodic traveling waves, it is sufficient to consider the set of Bloch parameters with $p \in [0, 1)$. Similar conditions apply in the y-direction, whose corresponding Bloch parameter is labeled q; we will combine these two parameters in a vector $\kappa = (p, q)^T$.