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Paraconformal structures, ordinary differential equations and totally geodesic manifolds

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ABSTRACT

We construct point invariants of ordinary differential equations of arbitrary order that generalise the Tresse and Cartan invariants of equations of order two and three, respectively. The vanishing of the invariants is equivalent to the existence of a totally geodesic paraconformal structure which consists of a paraconformal structure, an adapted $GL(2,\mathbb{R})$-connection and a two-parameter family of totally geodesic hypersurfaces on the solution space. The structures coincide with the projective structures in dimension 2 and with the Einstein–Weyl structures of Lorentzian signature in dimension 3. We show that the totally geodesic paraconformal structures in higher dimensions can be described by a natural analogue of the Hitchin twistor construction. We present a general example of Veronese webs that generalise the hyper-CR Einstein–Weyl structures in dimension 3. The Veronese webs are described by a hierarchy of integrable systems.

1. Introduction

A paraconformal structure, or a $GL(2,\mathbb{R})$-structure, on a manifold M is a smooth field of rational normal curves in the tangent bundle TM. The structures have been investigated since the seminal paper of Bryant [1] who has related the geometry of four-dimensional $GL(2,\mathbb{R})$-structures to the contact geometry of ordinary differential equations (ODEs) of order four and consequently constructed examples of spaces with exotic holonomies. The result of Bryant can be seen as a generalisation of the paper of Chern [2] who has proved that the conformal Lorentzian metrics on three-dimensional manifolds can be obtained from ODEs of order three (see also [3]). The higher dimensional cases have been treated by many authors, for example in [4–7]. It is proved that the solution space of an ODE has a canonical paraconformal structure if and only if the ODE satisfies the Wünschmann condition, i.e. certain contact invariants vanish.

In the present paper we consider paraconformal structures admitting the following additional structure: an adapted connection ∇ and a 2-parameter family of hyper-surfaces totally geodesic with respect to ∇. The structures will be referred to as the *totally geodesic paraconformal structures*. The structures are very well known in low dimensions. Indeed, in dimension 2 the structures coincide with the projective structures [8] and in dimension 3 the structures coincide with the Einstein–Weyl structures of Lorentzian signature [9,10]. A unified approach to the projective structures on a plane and to the three-dimensional Einstein–Weyl structures was given in the complex setting by Hitchin [11] in terms of a twistor construction. Much earlier, it was proved by E. Cartan that in both cases the geometry is related to the point geometry of ODEs [12,13]. The solution space of an ODE of order 2 or 3, respectively, has a canonical projective structure or an Einstein–Weyl structure, respectively, if and only if the Tresse invariant [14,15] or the Cartan and the Wünschmann [13,16] invariants, respectively, vanish.

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Our first aim in the present paper is to present a unified approach to the Tresse invariant of second order ODEs and to the Cartan invariant of third order ODEs. The approach generalises to higher order ODEs and as a result we introduce new point invariants of ODEs of arbitrary order. The second aim of the present paper is to analyse the geometry of the totally geodesic paraconformal structures. Finally we consider a general example based on special families of foliations, called Veronese webs, introduced by Gelfand and Zakharevich [17] in connection to bi-Hamiltonian structures on odd dimensional manifolds.

Our first new result is Theorem 3.2 that gives a characterisation of those paraconformal structures that can be constructed from ODEs. This result concerns the contact geometry of ODEs and, in a sense, completes results of [4–6]. Sections 4–9 concern point geometry of ODEs and are the core of the paper. In particular Theorems 6.1 and 7.1 provide new approach to the Tresse and Cartan invariants of ODEs of order 2 and 3 and give new, more simple, formulae for the invariants. Theorems 8.1 and 9.1 generalise the Tresse and Cartan invariants to higher dimensions.

Section 5 is devoted to a natural generalisation of the Hitchin twistor construction. The Hitchin construction involves a two-dimensional manifold and a curve with a normal bundle $O(1)$ or $O(2)$. Clearly one can consider curves with normal bundles $O(k), k > 2$. We argue that so-obtained structures correspond to higher-dimensional totally geodesic paraconformal structures. This should be compared to [18] where the authors are interested in torsion-free connections. On contrary, generic totally geodesic paraconformal structures considered in the present paper have non-trivial torsion. In the Hitchin's paper there is no construction of invariants on the side of ODEs. We concentrate on this issue in the present paper and our invariants characterise those equations for which the solutions are curves with self intersection number k.

Section 10 is devoted to the Ricci curvature tensor of a totally geodesic paraconformal connection. We prove that the symmetric part of the Ricci curvature tensor is a section of the bundle of symmetric 2-tensors annihilating all null directions of the structure. In dimension 3 the condition is equivalent to the Einstein–Weyl equation.

The last Section of the paper is devoted to Veronese webs [17,19,20]. We show that any Veronese web defines a totally geodesic paraconformal structure such that the associated twistor space fibres over $\mathbb{R}P^1$. In particular, Veronese webs in dimension 3 give an alternative description of the hyper-CR Einstein–Weyl structures [9,21,22]. We prove that in the general case the Veronese webs, or equivalently the totally geodesic paraconformal structures such that the corresponding twistor space fibres over $\mathbb{R}P^1$, are in a one to one correspondence with the solutions to the system

$$(a_i - a_j)\partial_0 w \partial_i \partial_j w + a_j \partial_i w \partial_j \partial_0 w - a_i \partial_j w \partial_i \partial_0 w = 0, \quad i, j = 1, \dots, k, \tag{1}$$

where a_i are distinct constants and $w\colon \mathbb{R}^{k+1} \to \mathbb{R}$. In this way we give a geometric meaning to the hierarchy of integrable systems introduced in [21].

Another applications of our results to the Veronese webs include: a construction of the canonical connections for the Veronese webs (Theorem 11.1) and a local characterisation of the flat Veronese webs in terms of the torsion of the canonical connection (Corollary 11.3). Moreover, we give new, elementary proof of the so-called Zakharevich conjecture [19] (Corollary 11.4). All these results translate to bi-Hamiltonian structures via the Gelfand–Zakharevich reduction [17].

2. Paraconformal structures and connections

Let M be a manifold of dimension $k+1$. A paraconformal structure on M is a vector bundle isomorphism

$$TM \simeq \underbrace{S \odot S \odot \cdots \odot S}_{k}$$

where S is a rank-two vector bundle over M and $\odot$ denotes the symmetric tensor product. It follows that any tangent space T_xM is identified with the space of homogeneous polynomials of degree k in two variables. The natural action of $GL(2,\mathbb{R})$ on S extends to the irreducible action on TM and reduces the full frame bundle to a $GL(2,\mathbb{R})$-bundle. Therefore the paraconformal structures are sometimes called $GL(2,\mathbb{R})$-geometries. We refer to [1,5] for more detailed descriptions of the paraconformal structures.

A paraconformal structure defines the following cone

$$C(x) = \{v \odot \cdots \odot v \mid v \in S(x)\} \subset T_xM$$

at each point $x \in M$ and it is an easy exercise to show that the field of cones $x \mapsto C(x)$ defines the paraconformal structure uniquely. If a basis e_0, e_1 in $S(x)$ is chosen then any $v \in S(x)$ can be written as $v = se_0 + te_1$ and then

$$C(x) = \{s^k V_0 + s^{k-1} t V_1 + \cdots + t^k V_k \mid (s,t) \in \mathbb{R}^2\}$$

where $V_i = \binom{k}{i} e_0^{\odot k-i} \odot e_1^{\odot i}$. We shall denote

$$V(s,t) = s^k V_0 + s^{k-1} t V_1 + \cdots + t^k V_k$$

and refer to the vectors as null vectors. The cone $C(x)$ defines a rational normal curve $(s : t) \mapsto \mathbb{R}V(s,t)$ of degree k in the projective space $P(T_xM)$. Sometimes, for convenience, we will use an affine parameter $t = (1 : t)$ and denote $V(t) = V_0 + tV_1 + \cdots + t^k V_k$. Derivatives of $V(t)$ with respect to t will be denoted $V'(t)$, $V''(t)$ etc. Let us stress that