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On homogeneous Einstein (α, β)-metrics

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ABSTRACT

In this paper, we study homogeneous Einstein (α, β)-metrics. First, we deduce a formula for Ricci curvature of a homogeneous (α, β)-metric. Based on this formula, we obtain a sufficient and necessary condition for a compact homogeneous (α, β)-metric to be Einstein and with vanishing S-curvature. Moreover, we prove that any homogeneous Ricci flat (α, β) space with vanishing S-curvature must be a Minkowski space. Finally, we consider left invariant Einstein (α, β)-metrics on Lie groups with negative Ricci constant. Under some appropriate conditions, we show that the underlying Lie groups must be two step solvable. We also present a more convenient sufficient and necessary condition for the metric to be Einstein in this special case.

1. Introduction

It is important to study Einstein manifolds in Riemann–Finsler geometry. A Finsler metric $F(x, y)$ on an n dimensional manifold M is called an Einstein metric if there exists a smooth function $\lambda(x)$ on M such that

$$\mathrm{Ric}(x, y) = \lambda(x)F^2(x, y).$$

In many occasions, S. S. Chern had asked the following problem openly:

Does every smooth manifold admit an Einstein Finsler metric?

This problem is extremely involved and remains open. However, the problem has motivated great interest of geometers and led to many results on Einstein Finsler metrics on manifolds. Up to now, most known Einstein Finsler metrics are either of Randers type or Ricci flat; see for example [1–7].

One effective approach to the above problem is to consider some special Finsler metrics. In this direction, invariant Einstein Finsler metrics on homogeneous manifolds are very interesting; see [8] for some results on homogeneous Einstein–Randers metrics.

In this paper, we consider homogeneous (α, β)-metrics. (α, β)-metrics form an important class of Finsler metrics. They are defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i(x)y^i$ in the form $F = \alpha\phi(\frac{\beta}{\alpha})$. It has been proved that $F = \alpha\phi(\frac{\beta}{\alpha})$ is a positive definite Finsler metric with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad |s| \le \rho < b_0. \tag{1.1}$$

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The Randers metric $F = \alpha + \beta$ is just the special (α, β)-metric with $\phi(s) = 1 + s$. The function ϕ is called of Randers type if there exist constants k_1, k_2, k_3 such that $\phi(s) = k_1\sqrt{1 + k_2 s^2} + k_3 s$. It is easy to see that F is a Randers metric if and only if ϕ is of Randers type.

In [9], L. Zhou first gave some formulas of Riemann curvature and Ricci curvature for (α, β)-metrics. Later, X. Cheng, Z. Shen and Y. Tian [3] found some errors in his formulas. They then present the correct formulas of Riemann curvature and Ricci curvature for (α, β)-metrics. They also show that if $\phi(s)$ is a polynomial in s, then the (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is Einstein if and only if it is Ricci flat.

In the present paper, based on the formula of Ricci curvature for (α, β)-metrics in [3], we give a formula of Ricci curvature for homogeneous (α, β)-metrics. Using this formula, we find a necessary condition related to ϕ for F to be Einstein. Then we show that if ϕ is normal (see Definition 3.4 and Proposition 3.5), then any compact homogeneous Einstein (α, β)-metric has vanishing S-curvature. Moreover, we obtain a sufficient and necessary condition for a compact homogeneous (α, β)-metric to be Einstein and with vanishing S-curvature. We also consider left invariant Einstein (α, β)-metrics on Lie groups with negative Ricci constant. Under the condition that ϕ is normal, we prove that the underlying Lie groups must be two step solvable. Meanwhile, we find a sufficient and necessary condition on ϕ such that any left invariant (α, β)-metric on such Lie groups is Einstein.

2. Preliminaries

In this section we recall briefly some known facts about Finsler spaces, for details, see [10,11,3,12]. Recall that a Minkowski norm on V is a real function F on V which is smooth on $V\backslash\{0\}$ and satisfies the following conditions:

(1) $F(u) \geq 0, \forall u \in V$;
(2) $F(\lambda u) = \lambda F(u), \forall \lambda > 0$;
(3) For any basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of V, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = y^i\varepsilon_i$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right)$$

is positive-definite at any point of $V\backslash\{0\}$.

A Finsler metric on a smooth manifold M is a function $F : TM \to [0, +\infty)$ which is C^∞ on the slit tangent bundle $TM\backslash\{0\}$ and whose restriction to any tangent space T_xM, $x \in M$ is a Minkowski norm.

Every Finsler metric induces a spray $\mathbf{G}$ on M defined by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i},$$

where

$$G^i(y) = \frac{1}{4}g^{il}\left[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\right].$$

$\mathbf{G}$ is a globally defined vector field on TM. The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a nonzero vector $y \in T_xM - \{0\}$, the Riemann curvature $R_y : T_xM \to T_xM$ is a linear map defined by

$$R_y(u) = R^i_k(y)u^k\frac{\partial}{\partial x^i}, \quad u = u^i\frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i\partial G^j}{\partial y^j\partial y^k}.$$

The trace of the Riemann curvature R_y is a scalar function Ric on TM defined by

$$\mathrm{Ric}(y) = \mathrm{tr}(R_y),$$

which is called the Ricci curvature of (M, F).

We now recall the notion of S-curvature of a Finsler space. It is a quantity to measure the rate of change of the volume form of a Finsler space along geodesics. S-curvature is a non-Riemannian quantity, or in other words, any Riemannian manifold has vanishing S-curvature. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{v_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i v_i) < 1\}},$$