Physica D 298–299 (2015) 1–12

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Arithmetic exponents in piecewise-affine planar maps

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ARTICLE INFO

Article history:
Received 6 January 2015
Accepted 27 January 2015
Available online 4 February 2015
Communicated by I. Melbourne

Keywords:
Piecewise-affine map
Arithmetic complexity
Arithmetic dynamics

ABSTRACT

We consider the growth of some indicators of arithmetical complexity of rational orbits of (piecewise) affine maps of the plane, with rational parameters. The exponential growth rates are expressed by a set of exponents; one exponent describes the growth rate of the so-called logarithmic height of the points of an orbit, while the others describe the growth rate of the size of such points, measured with respect to the p-adic metric. Here p is any prime number which divides the parameters of the map. We show that almost all the points in a domain of linearity (such as an elliptic island in an area-preserving map) have the same set of exponents. We also show that the convergence of the p-adic exponents may be non-uniform, with arbitrarily large fluctuations occurring arbitrarily close to any point. We explore numerically the behaviour of these quantities in the chaotic regions, in both area-preserving and dissipative systems. In the former case, we conjecture that wherever the Lyapunov exponent is zero, the arithmetical exponents achieve a local maximum.

1. Introduction

This paper is concerned with the analysis of indicators of arithmetical complexity of points of rational orbits of affine and piecewise affine planar maps. We present a combination of rigorous results and numerical experiments connecting the exponential growth rate of certain arithmetical functions to the dynamics on a divided phase space, where regular and irregular motions co-exist (see Fig. 1). Our aim is to complement existing quantitative measures of irregularity of motion – Lyapunov exponents, entropies – with measures of arithmetical complexity. Quantities of this type – the so-called heights – are well-established in diophantine geometry, and recently similar constructs (algebraic and arithmetical entropies) have been introduced in dynamics in the context of rational maps—see [1,2] and references therein. In particular, there is a fairly complete theory of heights for polynomial automorphisms [1, Section 7.1]. In a similar vein, the integrability criteria for discrete-time dynamical systems [3] have been extended to include tests of algebraic and arithmetical origin. Among the latter, the notion of diophantine integrability has recently been suggested, based on the slow (sub-exponential) growth of heights [4].

We are interested in monitoring the arithmetical complexity of the points of orbits of piecewise affine maps $F : \mathbb{Q}^2 \to \mathbb{Q}^2$ (notated $F : \mathbb{A}^2(\mathbb{Q}) \to \mathbb{A}^2(\mathbb{Q})$ when the distinction between affine and projective phase space is important). These maps feature highly complex dynamics from minimal ingredients, and the literature devoted to them is substantial, see, e.g., [5–13]. The increase in complexity of the iterates of a piecewise affine map derives solely from the growth of the coefficients, since the degree remains the same; these are the growth rates of interest to us. By contrast, for the iterates of polynomials and rational functions of degree greater than one, the growth of the degree is a preferred indicator of complexity—see for example [14–17,2].

The simplest measure of the complexity of a rational number $x = m/n$ is its *height* $H(x)$, defined as [1, Chapter 3]

$$H(m/n) = \max(|m|, |n|) \quad \gcd(m, n) = 1. \tag{1}$$

The notions of size and height are extended to two dimensions as follows

$$\|z\| = \max(|x|, |y|) \qquad H(z) = \max(H(x), H(y)) \quad z = (x, y). \tag{2}$$

The height will typically grow exponentially along orbits, so we define an allied quantity, the *arithmetic exponent* of the point z:

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log H(F^t(z)) \tag{3}$$

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http://dx.doi.org/10.1016/j.physd.2015.01.007
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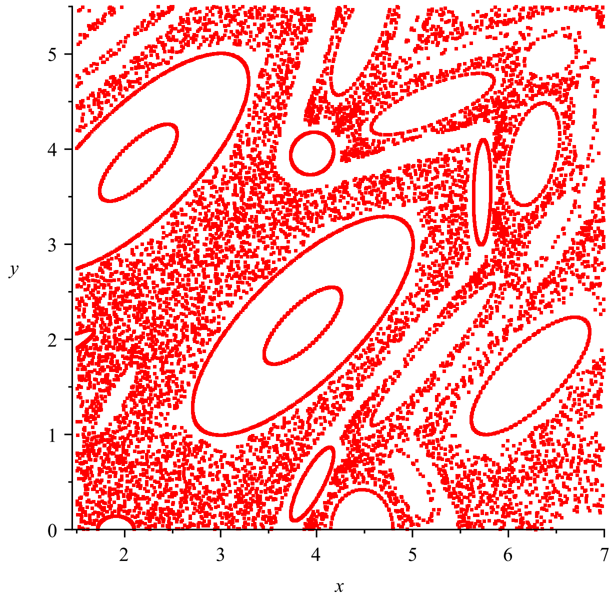

Fig. 1. Phase portrait of the area-preserving map F defined in Eq. (11), with f given in (12) and $d = 1$, showing a mixture of regular orbits on island chains and chaotic orbits.

if the limit exists. We see that the arithmetic exponent of a point z is the average order of the so-called logarithmic height $\log(H(z))$ of the images of this point. Since $\lambda(z) = \lambda(F(z))$, the arithmetic exponent is a property of an orbit. If z is a (pre)-periodic point, then $H(F^t(z))$ is bounded, so that $\lambda(z) = 0$ (as long as the orbit of z does not go through the origin).

The quantity $\lambda(z)$ is closely related to the *arithmetic entropy* introduced by Silverman [2], where the limit (3) is replaced by a lim sup.

Further indicators of complexity are defined by means of the p-adic absolute value $|\cdot|_p$, where p is a prime number. (For background reference on p-adic numbers, see [18].) Let the *order* $\nu_p(m)$ of an integer m be the largest non-negative integer k such that p^k divides m, with $\nu(0) = \infty$. This definition is extended to the rational numbers $r = m/n$ by letting $\nu_p(r) = \nu_p(m) - \nu_p(n)$ (the value of this expression does not depend on m and n being co-prime). Finally, we define

$$|r|_p = p^{-\nu_p(r)}.$$

The function $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}$ has the properties of the ordinary absolute value, with the triangular inequality replaced by the stronger ultrametric inequality

$$|x+y|_p \leqslant \max(|x|_p, |y|_p) \quad \text{or} \quad \nu_p(x+y) \geqslant \min(\nu_p(x), \nu_p(y)) \quad (4)$$

where equality holds if $|x|_p \neq |y|_p$ (or $\nu_p(x) \neq \nu_p(y)$). We shall be using the estimate

$$\nu_p(n) \leqslant \frac{\log n}{\log p} \qquad n \geqslant 1. \quad (5)$$

The following identity connects the various absolute values over $\mathbb{Q}$:

$$\forall x \in \mathbb{Q} \setminus \{0\}, \qquad |x| \prod_p |x|_p = 1 \quad (6)$$

where the product is taken over all primes. Only finitely many terms of this product are different from 1; they correspond to the prime divisors of the numerator and the denominator of x.

In two dimensions we use the quantities

$$\|z\|_p = \max(|x|_p, |y|_p) \qquad \nu_p(z) = \min(\nu_p(x), \nu_p(y)). \quad (7)$$

The norm $\|\cdot\|_p$ and valuation ν_p can be shown to satisfy the ultrametric inequalities analogous to (4), respectively, with equality holding if the two terms have distinct size. Next we define the analogue of (3), namely the *p-adic* (*arithmetic*) *exponent* $\lambda_p(z)$ of the initial point z of an orbit:

$$\lambda_p(z) = \lim_{t\to\infty} -\frac{1}{t}\, \nu_p(F^t(z)). \quad (8)$$

Comparing (8) with (3), we note that the function ν_p is already logarithmic, and that there is no need of considering separately numerator and denominator, since the prime p will appear only in one of them.

The functions λ and λ_p should be compared with the so-called *canonical height* defined for morphisms of degree greater than one [1, Chapter 3]. In this case, in place of (1) one defines

$$\hat{H}(m/n) = \max(|m|, |n|) \prod_p \max(|m|_p, |n|_p)$$

and then one lets

$$\hat{h}(x) = \lim_{t\to\infty} \frac{1}{\deg(F)^t} \log H(F^t(x))$$

where $\deg(F) > 1$ is the degree of F. The height $\hat{h}$ behaves nicely under iteration: $\hat{h}(F(x)) = \deg(F)\hat{h}(x)$. It measures the average rate of growth of the degree of F, collecting contributions from all absolute values. In our case, we have kept the contributions from the various primes separate (as in the so-called *local canonical heights*) because they contain valuable information about the dynamics.

The height may be used to characterise generic properties of rational points. To this end, we consider the set $\mathcal{B}_N$ of points in $\mathbb{Q}^2$ whose height is at most N:

$$\mathcal{B}_N = \{z \in \mathbb{Q}^2 : H(z) \leqslant N\}. \quad (9)$$

This set is finite. Indeed if $H(m/n) \leqslant N$, then $H(-m/n)$, $H(\pm n/m) \leqslant N$, and we deduce that

$$\#\mathcal{B}_N = \left(3 + 4\sum_{k=2}^{N} \phi(k)\right)^2 \sim \frac{12^2}{\pi^4} N^4 \quad (N \to \infty)$$

where ϕ is Euler's function [19, Section 5.5] and where we have used the estimate $\sum_{k=1}^{N} \phi(k) \sim 3N^2/\pi^2$ (see [19, Theorem 330] and also [1, p. 135]). Half of the elements of B_N lie within the square $\|z\| \leqslant 1$, where they approach a uniform distribution (because the Farey sequence has that property [20,21]); the other half lie outside the square, and they are obtained from the points inside the square by an inversion. Thus the limiting distribution of points of bounded height approaches a smooth limit on sufficiently regular bounded sets.

Let us now consider a set A such that $A \subset X \subset \mathbb{Q}^2$, where X is some ambient set (possibly the whole of $\mathbb{Q}^2$). The density $\mu(A)$ of A (in X) with respect to $\mathcal{B}_N$ is given by

$$\mu(A) = \lim_{N\to\infty} \frac{\#(A \cap \mathcal{B}_N)}{\#(X \cap \mathcal{B}_N)} \quad (10)$$

if the limit exists.[1] If $\mu(A) = 1$, then we say that A is 'generic', or that the defining property of A holds 'almost everywhere' (in X). For example, the rational points on a smooth curve on the plane have zero density and hence are non-generic.

[1] For this it suffices to require that the closure of the boundary of A has zero measure (Jordan measurability).