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Tracking pattern evolution through extended center manifold reduction and singular perturbations

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ABSTRACT

In this paper we develop an extended center manifold reduction method: a methodology to analyze the formation and bifurcations of small-amplitude patterns in certain classes of multi-component, singularly perturbed systems of partial differential equations. We specifically consider systems with a spatially homogeneous state whose stability spectrum partitions into eigenvalue groups with distinct asymptotic properties. One group of successive eigenvalues in the bifurcating group are widely interspaced, while the eigenvalues in the other are stable and cluster asymptotically close to the origin along the stable semi-axis. The classical center manifold reduction provides a rigorous framework to analyze destabilizations of the trivial state, as long as there is a spectral gap of sufficient width. When the bifurcating eigenvalue becomes commensurate to the stable eigenvalues clustering close to the origin, the center manifold reduction breaks down. Moreover, it cannot capture subsequent bifurcations of the bifurcating pattern. Through our methodology, we formally derive expressions for low-dimensional manifolds exponentially attracting the full flow for parameter combinations that go beyond those allowed for the (classical) center manifold reduction, i.e. to cases in which the spectral gap condition no longer can be satisfied. Our method also provides an explicit description of the flow on these manifolds and thus provides an analytical tool to study subsequent bifurcations. Our analysis centers around primary bifurcations of transcritical type – that can be either of co-dimension 1 or 2 – in two- and three-component PDE systems. We employ our method to study bifurcation scenarios of small-amplitude patterns and the possible appearance of low-dimensional spatio-temporal chaos. We also exemplify our analysis by a number of characteristic reaction–diffusion systems with disparate diffusivities.

1. Introduction

The analysis of pattern formation in evolutionary partial differential equations is directly linked to dynamical systems bifurcation theory. At the onset of patterns, a '*trivial state*' becomes spectrally unstable as a control or bifurcation parameter, R, passes through a critical value $R_{c,1}$. Typically, a '*small amplitude pattern*' bifurcates from this state. When the evolution equation is defined on a bounded domain Ω and the associated spectrum consists of discrete eigenvalues, the very first step in the onset of pattern formation can be studied by a center manifold reduction (CMR). For values of R sufficiently close to $R_{c,1}$, the dynamics of the full infinite-dimensional system can be reduced to the dynamics on an exponentially attracting low-dimensional center manifold, by virtue of the existence of a spectral gap between the first eigenvalue(s) crossing the imaginary axis and all other, stable eigenvalues. The presence of this gap makes the analysis of the onset of pattern formation completely equivalent to the study of bifurcations in finite-dimensional dynamical systems (for instance, [1–3]). Indeed, the small amplitude patterns that originate in this mechanism relate, in general, directly to the standard codimension 1 bifurcations (saddle–node, transcritical, pitchfork and Hopf): the associated center manifolds are 1- or 2-dimensional.

The center manifold reduction is only valid for R '*sufficiently close*' to the – first – critical value $R_{c,1}$, so that the spectral gap is sufficiently wide. However, in perhaps all examples of pattern forming systems, the pattern originating at $R_{c,1}$ undergoes a next bifurcation at some value $R_{c,2}$ of R et cetera. In other words, the first bifurcation at onset is followed by a secondary one at $R_{c,2}$. Since this latter concerns the bifurcating pattern and not the trivial state it bifurcated from, it cannot be directly studied through the

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spectral decomposition for that state. One now needs, instead, stability properties of the pattern bifurcating at $R_{c,1}$. Generally speaking, this is an impossible task — especially for analytical studies of pattern evolution. To overcome that obstacle, formal and/or numerical methods have been developed that are based on spectral properties – eigenvalues and eigenfunctions – associated with the original, trivial background state.

Such secondary, tertiary, et cetera subsequent bifurcations cannot be described by CMR, simply because they do not occur in the reduced (center manifold) flow. Therefore, they take place for values of R violating the spectral gap condition. This is often observed in an explicit setting: the distance between the first, now unstable, eigenvalue and the imaginary axis becomes proportional to that between the next largest eigenvalue(s) and the same axis — note carefully that none of these next eigenvalues needs to destabilize for the secondary bifurcation to occur. In the terminology of applied mathematics and/or physics: one *must* account for the evolution of '*modes*' associated with these next eigenvalues and eigenfunctions, as these modes can no longer be '*slaved*' to the one that was first destabilized and that parameterizes the center manifold. In principle, then, studying the full flow through the spectral properties of the trivial state is possible, provided that one extends CMR to a higher-dimensional system by a Galerkin approach. In general, however, there is no '*next*' spectral gap in that extended spectral problem: *all* next eigenvalues are typically commensurable. Accordingly, there is no telling a priori *how many* modes must be accounted for in this extended center manifold Galerkin reduction — certainly not from the analytic point of view. See, for instance, [4] and references therein for a practical study centering on these issues.

Presently, we develop analytical (and asymptotic) extensions of classical CMR. We describe the onset of pattern formation by means of low-dimensional systems governing the dynamics of the full evolutionary system for parameter values violating the spectral gap condition. We term the process by which we derive such simplified systems *extended center manifold reduction (ECMR)*. Our most generic results concern the extension of the 1-dimensional CMR associated with a transcritical bifurcation to an explicit 2-dimensional flow on an exponentially attracting 2-dimensional (local) manifold. We also present explicit classes of systems with codimension 1 bifurcations where this extended center manifold is 3- or 4-dimensional.

An earlier version of this method was developed in the context of a specific model problem, which concerned the emergence and evolution of localized spatio-temporal patterns in a non-local, coupled, phytoplankton-nutrient model in an oceanic setting,

$$\begin{cases}\omega_t = \varepsilon\omega_{xx} - 2\sqrt{\varepsilon}\,v\omega_x + (p(\omega,\eta,x)-\ell)\omega,\\ \eta_t = \varepsilon\left[\eta_{xx} + \ell^{-1}p(\omega,\eta,x)\omega\right];\end{cases} \tag{1.1}$$

this is a scaled version of the original model proposed in [5]. In (1.1), $\omega(x,t)$ and $\eta(x,t)$ denote a phytoplankton and a (translated) nutrient concentration; $x \in (0,1)$ measures ocean depth. The growth of the phytoplankton population is delimited by nutrient and light availability; since light is attenuated with depth and absorbed by phytoplankton, the term $p(\omega,\eta,x)$ is non-local in ω and depends explicitly on depth x. For more details on this model and its boundary conditions (BCs), see [5–7]. In realistic settings, $\varepsilon \approx 10^{-5}$ while all other parameters — v, ℓ and those entering $p(\omega,\eta,x)$ — can be considered $\mathcal{O}(1)$ with respect to ε [6]. Therefore, (1.1) is studied in [6,7] as a *singularly perturbed* system. The spectral problem associated with the stability of the trivial state $(\omega(x,t),\eta(x,t)) \equiv (0,0)$ — no phytoplankton, maximal and constant nutrient concentration — has two distinct sets of (real) eigenvalues: $\mu_m = \mathcal{O}(\varepsilon)$, $m \geq 1$, and $\lambda_n = \lambda_* + \tilde{\lambda}_n$, with $\tilde{\lambda}_n = \mathcal{O}(\varepsilon^{\frac{1}{3}})$ and $n \geq 1$; λ_* can be '*controlled*' by varying the parameters in (1.1), while $\mu_m < 0$ are parameter-independent and negative. In [6], it is shown through an asymptotic spectral analysis that the trivial state is destabilized by a transcritical bifurcation, at which λ_1 crosses zero. The associated eigenfunction has the strongly localized nature of a (stationary) *deep chlorophyll maximum* (DCM), the pattern playing a central role in the simulations and oceanic observations in [5].

In our terminology above, emergence of the deep chlorophyll maximum represents the onset of pattern formation, and it occurs as the product of the first bifurcation. For the parameter values considered in [5], the deep chlorophyll maximum only exists (as a stable, stationary pattern) in an asymptotically narrow strip of parameter space: the primary bifurcation is almost directly followed by a secondary, Hopf bifurcation through which emerges an oscillating deep chlorophyll maximum [6,7]. In fact, stationary deep chlorophyll maxima were not even recorded in the numerical simulations of [5] — the bifurcation scenario drawn there starts directly with the oscillating deep chlorophyll maximum and proceeds with period-doubling cascades and spatio-temporal chaos. In other parameter regimes, not a deep chlorophyll maximum, but a *benthic layer* — a localized maximum at ocean's bottom — marks pattern formation. Numerical simulations have not indicated a secondary bifurcation of the pattern in this regime. In [8], we analytically substantiate this phenomenon using the framework described in this treatise.

The predictions in [6] on the transcritical nature of trivial state destabilization were validated in [7], as a first step, by restricting analysis to the regime $0 < \lambda_1 = \mathcal{O}(\varepsilon^{\sigma})$ with $\sigma > 1$. In that case, there is a spectral gap driven by the proximity of that primary eigenvalue to the imaginary axis, $\lambda_1 \ll \min_{m\geq 1,n\geq 2}\{|\mu_m|,|\lambda_n|\} = \mathcal{O}(\varepsilon)$; the dynamics of system (1.1) can be reduced to a single amplitude ODE describing the transcritical bifurcation. As $\sigma \downarrow 1$ and λ_1 becomes $\mathcal{O}(\varepsilon)$ like the μ_m's, the spectral gap dissolves; modes associated with *all* (linearly stable!) μ_m-eigenvalues must now be taken into account. As a consequence, the 1-dimensional CMR is expanded dramatically into an a priori infinite-dimensional system. Analysis of that model is nevertheless possible and establishes the existence of a secondary Hopf bifurcation in (1.1), $\mathcal{O}(\varepsilon)$-close to the primary, transcritical one [7]. The existence of oscillating deep chlorophyll maxima follows.

In the present paper, we show that this surprising fact — that a secondary bifurcation becomes amenable to analysis by extending CMR beyond its classical region of validity — is not due to model specifics but intrinsically tied to the nature of the spectrum associated with the trivial background state. In general, our approach may be developed in the context of systems of the form

$$\frac{\partial}{\partial t}\begin{pmatrix}U\\V\end{pmatrix} = \begin{pmatrix}\mathcal{L} & 0\\ \varepsilon\mathcal{K} & \varepsilon\mathcal{M}\end{pmatrix}\begin{pmatrix}U\\V\end{pmatrix} + \begin{pmatrix}F(U,V;x)\\ \varepsilon G(U,V;x)\end{pmatrix}, \tag{1.2}$$

for a 'fast', unknown $U : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m_U}$ and a 'slow' $V : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m_V}$, with $m_U, m_V \geq 1$. The bounded spatial domain $\Omega \subset \mathbb{R}^n$ has a piecewise C^1 boundary $\partial\Omega$. The operators $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ are assumed linear, spatial, differential operators and boundary conditions guaranteeing well-posedness must apply. Several specific assumptions on the spectrum of $\mathcal{L}$ and $\mathcal{M}$ and the nonlinearities $F(U,V;x)$ and $G(U,V;x)$ must hold, we refer to [8] for more details. The aim of this paper is to present an exploration into the possible impact of the extended center manifold reduction approach. Therefore, we will mainly restrict our analysis to a strongly simplified version of (1.2), i.e. to models of the type

$$\begin{cases}U_t = \mathcal{L}U + \alpha U + F(U,V),\\ V_t = \varepsilon\left[\mathcal{L}V + \beta U + \gamma V + G(U,V)\right],\end{cases} \tag{1.3}$$

thus $\mathcal{K} = \beta$, and with a slight abuse of notation, $\mathcal{M} = \mathcal{L} + \gamma$ and the operator $\mathcal{L}$ in (1.2) will be replaced by $\mathcal{L} + \alpha$. The linear differential operator $\mathcal{L}$ in (1.3) — independent of ε — acts on $L^2(\Omega)$,