Journal of Geometry and Physics 102 (2016) 1–31

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Bäcklund transformations for Darboux integrable differential systems: Examples and applications

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ARTICLE INFO

Article history:
Received 28 October 2015
Accepted 15 December 2015
Available online 23 December 2015

MSC:
37K35
58A15
58J70
34A26

Keywords:
Bäcklund transformation
Darboux integrability
Exterior differential systems
Symmetry reduction

ABSTRACT

In this article we demonstrate a new symmetry based method for constructing Bäcklund transformations by finding explicit Bäcklund transformations between Darboux integrable systems. This results in a number of new examples of Bäcklund transformations which are quite different in character than those typically found in the literature. The relation between the intermediate integrals for Darboux integrable systems and the differential invariants of the Vessiot group is also illustrated. We then show that a well known class of Bäcklund transformations between a Darboux integrable Monge–Ampère system and the wave equation always arises by this method. The results of this paper build upon the presentation of Darboux integrable systems as quotients of differential systems by symmetry groups.

1. Introduction

This article is a continuation of our paper *Bäcklund Transformations for Darboux Integrable Differential Systems* [1]. There we established a general group-theoretical approach to the construction of Bäcklund transformations and we also showed how this construction can be applied to construct Bäcklund transformation between equations which are Darboux integrable. In this paper we demonstrate this theory with a number of detailed examples and new applications.

We adopt the coordinate-free formulation of Bäcklund transformations provided by the differential-geometric setting of exterior differential systems (EDS) [2]. From this viewpoint, two differential systems $\mathcal{I}_1 \subset \Omega^*(N_1)$ and $\mathcal{I}_2 \subset \Omega^*(N_2)$, defined on manifolds N_1 and N_2, are said to be related by a Bäcklund transformation if there exists a differential system $\mathcal{B} \subset \Omega^*(N)$ on a manifold N and maps

$$\begin{array}{ccc} & (\mathcal{B}, N) & \\ {}^{\mathbf{p}_1}\swarrow & & \searrow^{\mathbf{p}_2} \\ (\mathcal{I}_1, N_1) & & (\mathcal{I}_2, N_2) \end{array} \tag{1.1}$$

which define $\mathcal{B}$ as integrable extensions for both $\mathcal{I}_1$ and $\mathcal{I}_2$. The main result from [1] on the construction of Bäcklund transformations by symmetry group reduction is the following theorem.

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http://dx.doi.org/10.1016/j.geomphys.2015.12.005
0393-0440/

Theorem A. *Let $\mathcal{I}$ be a differential system on a manifold M with Lie symmetry groups G_1 and G_2. Let H be a common subgroup of G_1 and G_2 and assume that the actions of G_1, G_2 and H are all regular on M. Then the orbit projection maps $\mathbf{p}_1 : M/H \to M/G_1$ and $\mathbf{p}_2 : M/H \to M/G_2$ are smooth surjective submersions and*

$$
\begin{array}{ccccc}
 & & (\mathcal{I}, M) & & \\
 & \swarrow^{\mathbf{q}_{G_1}} & \downarrow^{\mathbf{q}_H} & \searrow^{\mathbf{q}_{G_2}} & \\
 & & (\mathcal{I}/H, M/H) & & \\
 & \swarrow^{\mathbf{p}_1} & & \searrow^{\mathbf{p}_2} & \\
(\mathcal{I}/G_1, M/G_1) & & & & (\mathcal{I}/G_2, M/G_2)
\end{array}
\tag{1.2}
$$

is a commutative diagram of EDS. Furthermore, if the actions of G_1 and G_2 are transverse to $\mathcal{I}$, then the maps in (1.2) are all integrable extensions and the diagram

$$
\begin{array}{ccc}
 & (\mathcal{B} = \mathcal{I}/H,\ N = M/H) & \\
\swarrow^{\mathbf{p}_1} & & \searrow^{\mathbf{p}_2} \\
(\mathcal{I}_1 = \mathcal{I}/G_1,\ N_1 = M/G_1) & & (\mathcal{I}_2 = \mathcal{I}/G_2,\ N_2 = M/G_2)
\end{array}
\tag{1.3}
$$

defines $\mathcal{B} = \mathcal{I}/H$ as a Bäcklund transformation between $\mathcal{I}_1 = \mathcal{I}/G_1$ and $\mathcal{I}_2 = \mathcal{I}/G_2$.

We remark that the maps $\mathbf{p}_i$ in the diagram (1.3) are in general **not** group quotients.

This paper is organized as follows. Section 2 provides a basic review of the definitions and theory required to construct diagram (1.2). In Section 3 we then use Theorem A and the methods from Section 2 to give explicit examples of Bäcklund transformations in local coordinates. In each example we shall begin with a differential system $\mathcal{I}$ given as a direct sum $\mathcal{K}_1 + \mathcal{K}_2$ on a product manifold $M_1 \times M_2$. We define a group action H acting diagonally on $M_1 \times M_2$ which is a symmetry group of $\mathcal{I}$ and which acts transversely to $\mathcal{I}$. We then calculate the reduced differential system $\mathcal{B} = \mathcal{I}/H$ in local coordinates. We then pick two more Lie symmetry groups G_1 and G_2 of $\mathcal{I}$, with $H \subset G_1 \cap G_2$, and calculate the reduced differential systems $\mathcal{I}_1 = \mathcal{I}/G_1$ and $\mathcal{I}_2 = \mathcal{I}/G_2$. The orbit projection maps $\mathbf{p}_i : (\mathcal{B}, N) \to (\mathcal{I}_i, N_i)$, which define the sought-after Bäcklund transformation, are also given in local coordinates.

The examples in Section 3 satisfy a property called Darboux integrability. In Section 4.2 we show how Darboux integrable systems are constructed using symmetry reduction as well as how to compute the fundamental invariants for these systems using symmetry reduction (Theorems 4.5 and 4.6). In Section 5 we revisit the examples from Section 3 from the perspective of Darboux integrability. The intermediate integrals are computed using Theorem 4.5. The fundamental invariant called the ***Vessiot algebra*** (or Vessiot group) is also determined, using Theorem 4.6, for each example.

Section 6 provides some basic theory relating the fundamental invariants of a Darboux integrable system to those of an integrable extension. Section 7 combines the theory from Section 6 and the geometry of the double fibration for a Bäcklund transformation to show that *every Bäcklund transformation constructed in* [2] *and* [3] *between hyperbolic Monge–Ampère systems and the hyperbolic Monge–Ampère system for the wave equation arises by symmetry reduction.* For Bäcklund transformation of this type, Theorem 7.8 shows the Vessiot algebra for $\mathcal{B}$ is isomorphic to a 2-dimensional subalgebra of the Vessiot algebra of $\mathcal{I}_2$. We give an explicit example of a Darboux integrable Monge–Ampére equation with $\mathfrak{so}(3, \mathbf{R})$ Vessiot algebra. Because $\mathfrak{so}(3, \mathbf{R})$ has no 2-dimensional subalgebras, Theorem 7.8 implies that $\mathcal{I}_2$ cannot admit such a Bäcklund transformation to the wave equation.

The proof of Theorem 7.8 uses a novel condition, called ***maximal compatibility*** (defined in Section 6), which insures that an integrable extension between Darboux integrable systems arises by symmetry reduction. It is our belief that this condition can be generalized to other settings which will broaden the scope of Theorem A as a mechanism underlying other Bäcklund transformations.

The calculations for this paper were performed using the Maple DifferentialGeometry package. The current release of the DifferentialGeometry package, as well as worksheets which support our results, can be downloaded from http://digitalcommons.usu.edu/dg/. The work by Ian Anderson was supported by the grant NSF-OCI-1148331 from the National Science Foundation.

2. Preliminaries

In this section we gather together a number of definitions and basic results on integrable extensions and reductions of exterior differential systems. These are needed to construct diagram (1.2) and hence Bäcklund transformations by Theorem A. Conventions and some additional results from [4] are used here, see also [5,1].