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On the asymptotic behavior of the Wigner transform for large values of Planck's constant

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ABSTRACT

We give an asymptotic expression for the Wigner transform of certain functions for large values of Planck's constant. We express our results with the help of Kazhdan constants depending on a compact set and a representation which are useful in the analysis of Kazhdan's property T. This allows us to show that for every compact set there exist functions whose Wigner transform have no zeros thereon. We also determine such functions explicitly.

1. Introduction

Let f be a square-integrable function on $\mathbb{R}^n$. By definition its Wigner transform (or distribution) is

$$Wf\,(x,p)=\left(\frac{1}{2\pi\hbar}\right)^n\int_{\mathbb{R}^n}e^{-\frac{i}{\hbar}p\cdot y}f\left(x+\tfrac{1}{2}y\right)\overline{f\left(x-\tfrac{1}{2}y\right)}\,\mathrm{d}y$$

for $x, p \in \mathbb{R}^n$ where $p\cdot y$ denotes the standard euclidean scalar product with $y \in \mathbb{R}^n$. The Wigner transform plays, in quantum mechanics, a role similar to that of a probability density in classical mechanics; in particular its marginals are the position and momentum probabilities densities for a quantum state. However, as opposed to what is the case for a bona fide probability, the Wigner transform can take negative values; this is in fact always the case unless f is a Gaussian (Hudson's theorem [1]). While it is customary, especially in semiclassical mechanics, to study the asymptotic behavior of Wf for $\hbar \to 0$, there are, to

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the best of the knowledge of the present authors, almost no results for large values of $\hbar$, which corresponds to what we call the "ultraquantum case". Only in [2] deformations of two parameters have been considered where the second parameter is $\beta = \frac{1}{kT}$ in relation to the Kubo–Martin–Schwinger boundary condition. The case that the temperature $T \to 0$ corresponds to the deformation parameter $\beta \to \infty$. The results in the present work will be achieved by using methods for the analysis of a deep and difficult result from representation theory, Kazhdan's property T.

The outline of the paper is as follows. We first recall a few facts about Kazhdan's property T. Then we draw consequences for the Heisenberg group and the Schrödinger representations. In the fourth section we apply this to obtain results for the Wigner–Moyal transform from time-frequency analysis. There we show the existence of certain functions for which the Wigner–Moyal transform has no zero on a compact subset. In the last section we compute such functions explicitly.

2. Kazhdan's property T

Kazhdan's property T is a representation theoretic property with remarkable applications. It was introduced in 1967 by D.A. Kazhdan [3]. Among its applications are the efficient construction of telecommunication networks [4], the mixing of random walks [5], the product replacement algorithm [6], etc. For an account see [7].

Let H be a topological group and R a strongly continuous unitary representation of H. Let $\varepsilon > 0$ and Q a compact subset of H. A vector f is called (Q, ε)-*invariant* if

$$\|R(w)f - f\| < \varepsilon \|f\| \quad \forall\, w \in Q.$$

The representation R is said to *almost have invariant vectors* if for every $\varepsilon > 0$ and every compact Q there exist (Q, ε)-invariant vectors. The group has *Kazhdan's property* T if every representation which almost has invariant vectors has in fact a non-zero invariant vector i. e. there exists an $f \neq 0$ such that $R(w)f = f$ for all $w \in H$.

This can also be formulated in a quantitative way. Let $Q \subset H$ then

$$\kappa(Q, R) = \inf_{\|f\|=1} \sup_{w \in Q} \|R(w)f - f\|$$

is called the *Kazhdan constant* with respect to Q and the representation R. The group H has Kazhdan's property T if and only if there exists a compact subset Q of H such that $\inf_R \kappa(Q, R) > 0$ where R ranges through all classes of representations which do *not* have invariant vectors. It is even enough to consider irreducible representations of H i. e. Kazhdan's property T is even equivalent to $\inf_R \kappa(Q, R) > 0$ where here R ranges through all classes of irreducible representations which are not trivial.

Note that a group has Kazhdan's property T if and only if the trivial representation is isolated in the so-called Fell topology. This topology is defined on the space of equivalence classes of irreducible representations, see e. g. [7].

3. Heisenberg group and Schrödinger representations

Let now $H = \mathbb{R}^{2n+1}$ be the *Heisenberg group* with group multiplication given by

$$(x, p, t)(x_0, p_0, t_0) = \left(x + x_0, p + p_0, t + t_0 + \tfrac{1}{2}(p \cdot x_0 - p_0 \cdot x)\right).$$

For $\hbar \in \mathbb{R} \setminus \{0\}$ let $\mathcal{R}_\hbar$ be the *Schr ödinger representation* on $L^2(\mathbb{R}^n)$ defined by

$$\mathcal{R}_\hbar(x, p, t) f(y) = \exp\left(\frac{2\pi i}{\hbar}\left(t + p \cdot y - \tfrac{1}{2} p \cdot x\right)\right) f(y - x)$$

which is irreducible.

Theorem 1. *In the Fell topology the Schrödinger representations converge to the trivial representation for $\hbar \to \infty$.*

For a proof see for example [8] or [9].

Theorem 2. *The Heisenberg group does not have property* T. *For any compact subset Q of H we have*

$$\lim_{\hbar \to \infty} \kappa(Q, \mathcal{R}_\hbar) = \lim_{\hbar \to \infty} \inf_{\|f\|_2=1} \sup_{w \in Q} \|\mathcal{R}_\hbar(w) f - f\|_2 = 0.$$

This follows from the previous result and the fact that the representations $\mathcal{R}_\hbar$ are irreducible. For this see again [8] or [9] or the proof in the last section.

Remark 3. The Heisenberg group is nilpotent and a non-compact nilpotent (also solvable, even amenable) group does not have property T. This already implies the weaker statement that the infimum is 0 instead of the limit. In the last section we give an elementary proof of the theorem also without appealing to property T.

While in the limit $\kappa(Q, \mathcal{R}_\hbar)$ converges to 0 in general $\kappa(Q, \mathcal{R}_\hbar) > 0$ for fixed $\hbar$.

Theorem 4. *Let Q be a compact subset of H and suppose $Q' \setminus (\{0\} \times \hbar\mathbb{Z}) \neq \emptyset$ where $Q' = \{[w, w_0] : w, w_0 \in Q\}$ with the commutator $[w, w_0] = w^{-1} w_0^{-1} w w_0$ of $w, w_0 \in H$ then $\kappa(Q, \mathcal{R}_\hbar) > 0$.*