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Analysis of non-homogeneous heat model with new trend of derivative with fractional order

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ABSTRACT

The model of nonlinear heat was generalized using the new trend of derivative with fractional order. The new definition of derivative with fractional order has no singular kernel thus allows a description of the variation on time or space from the lower to the upper boundaries within the space/time interval which the investigation is taken place for a given model. In detail, we presented the analysis of unique and existence of a solution for the nonlinear fractional equation. We present the derivation of a special solution using an iterative method.

1. Motivation

The purpose of inertial confinement fusion is to demonstrate that the thermal conductivity is toughly reliant on the temperature and the heat equation condition is nonlinear equation. The heat equation heads heat diffusion, in addition to other diffusive developments, for instance the particle diffusion or the prolongation of action potential in nerve cells [1-3]. Some significant mechanics problems are also ruled by mathematical equivalent of heat equation. The mathematical tool can also be employed to describe some phenomena ascending in finance, for example the Black–Scholes or the Ornstein–Uhlenberk processes [3]. The non-linear heat equations have also been used in image analysis. However, these models with the local derivative cannot fully and accurately describe the observed real world occurrence, for example in finance one would like to know what happen 3 days before, which is the idea of memory [4-7]. However, in many research papers found in the literature, it was with proof demonstrated that derivatives fractional order can be useful to enhance some model describe by the local derivative, in particular when the heat flow takes place within a medium with many layers, the novel derivative with fractional order introduced by Caputo and Fabrizio can be suitable to best describe such event [8-22]. Therefore, we analyze the nonlinear heat model with the new trend of fractional derivatives. The equation under study is given by

$$u_t = u_{xx} + \epsilon u^m \tag{1}$$

The next section is devoted to definitions and some properties of the recent introduced derivatives with fractional order.

2. New derivatives with fractional order

We present in this section the new definitions of derivative with fractional order.

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Definition 1. [8] Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$ then, the definition of the new Caputo fractional derivative is

$$D_t^\alpha(f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx, \quad (2)$$

where $M(\alpha)$ denotes a normalization function obeying $M(0) = M(1) = 1$ [8]. However, if the function does not belongs to $H^1(a,b)$ then, the derivative has the form

$$D_t^\alpha(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (f(t)-f(x))\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx \quad (3)$$

Remark. If $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty)$, $\alpha = \frac{1}{1+\sigma} \in [0,1]$, then the Eq. (2) assumes the form

$$D_t^\alpha(f(t)) = \frac{N(\sigma)}{\sigma}\int_a^t f'(x) \times \exp\left[-\frac{t-x}{\sigma}\right]dx, \quad N(0) = N(\infty) = 1 \ . \quad (4)$$

In addition we have

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \quad (5)$$

Definition 2. Let f be a function not necessary differential, let α be a real number such that $0 \le \alpha \le 1$, then the Caputo–Fabrizio derivative in Riemann–Liouville sense with order α is given as [13]

$$ {}_0^{AD}D_x^\alpha\{f(x)\} = \frac{1}{1-\alpha}\frac{d}{dx}\int_0^x f(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx. \quad (6)$$

If alpha is zero we have

$$ {}_0^{AD}D_x^0\{f(x)\} = \frac{d}{dx}\int_0^x f(x)dx = f(x).$$

Using the argument by Caputo and Fabrizio, we also have that when also goes to 1 we recover the first derivative.

Theorem 1. The Caputo–Fabrizio derivative with fractional order in Caputo sense is connected to the Caputo–Fabrizio derivative in Riemann–Liouville sense

$$\theta(\alpha){}_0^{CF}D_x^\alpha[f(x)] = \theta(\alpha){}_0^{AD}D_x^\alpha[f(x)] + f(0)\exp[-f(\alpha)x],$$
$$\theta(\alpha) = \frac{M(\alpha)}{1-\alpha}, \quad f(\alpha) = \frac{\alpha}{1-\alpha}. \quad (7)$$

Proof: By definition, we have the following [13]:

$$\theta(\alpha){}_0^{CF}D_x^\alpha[h(x)]$$
$$= \frac{d}{dx}\int_0^x h(t)\exp[-f(\alpha)(x-t)]dt$$
$$= h(x) - f(\alpha)\int_0^x h(t)\exp[-f(\alpha)(x-t)]dt$$
$$= h(x) - f(\alpha)\left\{\frac{h(x)}{f(\alpha)} - \frac{h(0)}{f(\alpha)}\exp[-f(\alpha)x] - \frac{1}{f(\alpha)}\int_0^x \frac{df(t)}{dt})\exp[-f(\alpha)(x-t)]dt\right\}$$
$$= \frac{h(0)}{f(\alpha)}\exp[-f(\alpha)x] + \int_0^x \frac{df(t)}{dt}\exp[-f(\alpha)(x-t)]dt$$
$$= {}_0^{AD}D_x^\alpha[f(x)] + \frac{h(0)}{f(\alpha)}\exp[-f(\alpha)x]. \quad (8)$$

3. Existence and uniqueness of a special solution

In this section, the analysis of the extended non-homogeneous heat model with new trend of derivative with fractional order is presented. Based upon our motivation in Section 2, the time derivative of Eq. (1) is replaced by the time-fractional derivative to obtain

$$ {}_0^{CF}D_t^\alpha u(x,t) = u_{xx}(x,t) + \epsilon u^m(x,t) \quad (9)$$

The above equation will be first converted to the Volterra equation by applying on both sides the associate integral to obtain

$$u(x,t) - u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{u_{xx}(x,t) + \epsilon u^m(x,t)\} + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \times\{u_{xx}(x,y) + \epsilon u^m(x,y)\}dy \quad (10)$$

For simplicity

$$u(x,t) = u(x,0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy \quad (11)$$

The operator K has Lipschitz condition providing that u has an upper bound. Thus if we have the upper bound then,

$$\| K(x,t,u) - K(x,t,v) \| = \| \{u_{xx}(x,t) - v_{xx}(x,t)\} + \epsilon\{u^m(x,t) - v^m(x,t)\} \| \quad (12)$$

Then,

$$\| K(x,t,u) - K(x,t,v) \| \le \| \{u_{xx}(x,t) - v_{xx}(x,t)\} \| + \epsilon \| u^m(x,t) - v^m(x,t) \| \quad (13)$$

where

$$\| u(x,t) \| = \max_{(x,t)\in X.T}|u(x,t)|$$

The first component of the right hand side can be evaluated using the Lipchitz condition of the operator derivative such that:

$$\| \{u_{xx}(x,t) - v_{xx}(x,t)\} \| \le \emptyset_1^2 \| u - v \| \quad (14)$$