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Biharmonic maps on tangent and cotangent bundles

E. Calviño-Louzao [a], E. García-Río [b,*], M. Sixto-Neira [b], M.E. Vázquez-Abal [b]

[a] Department of Mathematics, IES Ramón Caamaño, Muxía, Spain

[b] Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain

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ABSTRACT

We examine the harmonicity of some natural maps associated to the tangent and cotangent bundles, providing some new examples of proper biharmonic maps between pseudo-Riemannian manifolds.

1. Introduction

A major flaw in Riemannian geometry (as compared to other subjects) is a shortage of suitable kind of functions from one manifold to another that will compare their geometric properties. Generalizing the local isometries, totally geodesic maps preserve geodesics and hence they are suitable for comparing the local geometries of the domain and the range manifolds. However totally geodesic maps are too rigid and, for many purposes, harmonic maps are of major interest. As a generalization of harmonic maps, the study of biharmonic maps was suggested by Eells and Sampson [1]. While harmonic maps between compact Riemannian manifolds are defined as being critical points of the energy functional, biharmonic maps are critical points of the bienergy functional (the L^2-norm of the tension field).

During the last decades there has been a growing interest in the theory of biharmonic maps which can be divided in two main directions. The analytic point of view focuses on the study of biharmonic maps as solutions of a fourth order strongly elliptic semilinear PDE. However the geometric approach has been mainly devoted to the construction of examples and classification results. The latter was also considered in the pseudo-Riemannian case and the main purpose of this work is to contribute into this direction by constructing new examples of biharmonic maps.

Clearly any harmonic map is biharmonic but the converse is not true in general. Thus the task of constructing biharmonic maps which are not harmonic is an interesting one to pursue. One such construction has consisted in chosen suitable conformal deformations so that a map becomes biharmonic. In particular, it turns out that the existence problem for proper biharmonic submanifolds in pseudo-Riemannian manifolds often appears considerably different from its Riemannian counterpart, and the classification problems are more complicated. In this paper we support the above by showing

* Corresponding author.
E-mail addresses: estebcl@edu.xunta.es (E. Calviño-Louzao), eduardo.garcia.rio@usc.es (E. García-Río), marta.sixto.neira@gmail.com (M. Sixto-Neira), elena.vazquez.abal@usc.es (M.E. Vázquez-Abal).

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many examples of proper biharmonic maps between pseudo-Riemannian manifolds. All these maps are constructed from geometric data and their domain and/or range are the tangent or cotangent bundle of a given pseudo-Riemannian manifold. It is worth to emphasize that although the kind of maps we are interested in are not harmonic, their tension field is a non-zero null vector field for the corresponding pseudo-Riemannian metrics.

2. Harmonic and biharmonic maps

2.1. Harmonic maps

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. Define the *energy density of* φ by $e(\varphi) := \frac{1}{2}\|d\varphi\|^2$. The *energy functional* (assuming either that M is compact or that φ has compact support) is given by:

$$E(\varphi) := \int_M e(\varphi) dM,$$

where dM denotes the volume element of (M, g). In this setting, φ is said to be *harmonic* if it is a critical point of the energy functional $E(\varphi)$.

Let $\varphi^{-1}(TN)$ be the pull-back bundle. Then the Levi-Civita connections on TM and TN induce a connection ∇ in the bundle of one-forms on M with values in $\varphi^{-1}(TN)$. Then $\nabla d\varphi$ is a symmetric bilinear form on TM which is called the *second fundamental form* of φ. The trace of $\nabla d\varphi$ with respect to g is called the *tension field* of φ, and is denoted by $\tau(\varphi)$.

It turns out that the Euler–Lagrange equations corresponding to the critical points of the energy functional correspond to the vanishing of the tension field of φ, and hence the map φ is said to be *harmonic* if $\tau(\varphi) = 0$ and *totally geodesic* if the second fundamental form vanishes. (See [1] and [2,3] for more details and references.)

Now, let $U \subset M$ be a domain with coordinates $(x^1, \dots, x^m)$, $m = \dim M$ and $V \subset N$ be a domain with coordinates $(z^1, \dots, z^n)$, $n = \dim N$, such that $\varphi(U) \subset V$ and suppose that φ is locally represented by $z^\alpha = \varphi^\alpha(x^1, \dots, x^m)$, $\alpha = 1, \dots, n$. Then we have:

$$(\nabla d\varphi)^\gamma_{ij} = \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} + {}^h\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j}. \tag{1}$$

Here ${}^g\Gamma^k_{ij}$ and ${}^h\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of (M, g) and (N, h), respectively. So, φ is harmonic if and only if

$$\tau(\varphi) = \mathrm{trace}_g(\nabla d\varphi) = g^{ij}(\nabla d\varphi)^\gamma_{ij} \frac{\partial}{\partial z^\gamma} = 0. \tag{2}$$

2.2. Biharmonic maps

In analogy with the definition of harmonic maps, the *bienergy density* of a map $\varphi : (M, g) \longrightarrow (N, h)$ is defined by $e^2(\varphi) = \frac{1}{2}\|\tau(\varphi)\|^2$, and the corresponding *bienergy functional*:

$$E^2(\varphi) = \int_M e^2(\varphi) dM,$$

as a kind of L^2-norm of the tension field of φ. Now, a smooth map $\varphi : (M, g) \to (N, h)$ is called *biharmonic* if it is a critical point of the bienergy functional.

It now follows that the Euler–Lagrange equations for the critical points of the bienergy functional are equivalent to the vanishing of the *bitension field* $\tau^2(\varphi)$ given by (see for example [4–8] and references therein):

$$\tau^2(\varphi) = \mathrm{trace}_g(\nabla^2 d\varphi) = \mathrm{trace}_g\{({}^h\nabla^h\nabla - {}^h\nabla_{{}^g\nabla})\tau(\varphi) - {}^hR(\varphi_*, \tau(\varphi))\varphi_*\},$$

where the curvature tensor is taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

A coordinate expression of the bitension field is obtained in an analogous way as in the above as follows (see also [9]),

$$\begin{aligned}\tau^2(\varphi) &= \mathrm{trace}_g(\nabla^2 d\varphi)\\ &= g^{ij}\left\{\frac{\partial^2 \tau^\sigma}{\partial x^i \partial x^j} + \frac{\partial \tau^\alpha}{\partial x^j}\frac{\partial \varphi^\beta}{\partial x^i}{}^h\Gamma^\sigma_{\alpha\beta} + \frac{\partial}{\partial x^i}\left(\tau^\alpha \frac{\partial \varphi^\beta}{\partial x^j}{}^h\Gamma^\sigma_{\alpha\beta}\right)\right.\\ &\quad\left. + \tau^\alpha \frac{\partial \varphi^\beta}{\partial x^i}\frac{\partial \varphi^\rho}{\partial x^i}{}^h\Gamma^\nu_{\alpha\beta}{}^h\Gamma^\sigma_{\nu\rho} - {}^g\Gamma^k_{ij}\left(\frac{\partial \tau^\alpha}{\partial x^k} + \tau^\alpha \frac{\partial \varphi^\beta}{\partial x^k}{}^h\Gamma^\sigma_{\alpha\beta}\right) - \tau^\nu \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j}{}^hR^\sigma_{\beta\alpha\nu}\right\}\frac{\partial}{\partial z^\sigma},\end{aligned} \tag{3}$$

where $\tau(\varphi) = \tau^\sigma \frac{\partial}{\partial z^\sigma}$ is the coordinate expression of the tension field of φ and superindices denote the components $\varphi = (\varphi^1, \dots, \varphi^n)$. Moreover, ${}^g\Gamma$ and ${}^h\Gamma$ denote the Christoffel symbols of the Levi-Civita connections of (M, g) and (N, h), and ${}^hR$ is the curvature tensor of (N, h), where we follow the notation ${}^hR^\sigma(\partial_{z^\alpha}, \tau)\partial_{z^\beta} = \tau^\nu\, {}^hR^\sigma_{\beta\alpha\nu}$.