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Generalization of bi-Hamiltonian systems in $(3+1)$ dimension, possessing partner symmetries

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ABSTRACT

We study bi-Hamiltonian structure of a general equation which possesses partner symmetries. The general form of such second-order PDEs with four independent variables was determined in the paper Sheftel and Malykh (2009) on a classification of second-order PDEs which have this property. We apply Dirac's theory of constraints to this general equation. We formulate the equation in a two-component form and present the Lax pair of Olver–Ibragimov–Shabat type. Under some constraints imposed on constant coefficients of this equation, we obtain its bi-Hamiltonian structure. Therefore, by Magri's theorem it is a completely integrable bi-Hamiltonian system in $(3+1)$ dimensions. We also showed that with suitable choices of constant coefficients the equation is reduced to the well known integrable bi-Hamiltonian systems in $(3+1)$ dimension.

1. Introduction

We discover bi-Hamiltonian structure of a general scalar second-order PDE with four independent variables which possesses partner symmetries. The definition of partner symmetries [1] requires two conditions to be satisfied:

1. The symmetry condition for a given PDE (determining symmetries of the PDE) has the form of a two-dimensional divergence that implies the existence of a unique potential for each symmetry.
2. The potential of each symmetry is itself a symmetry of the PDE called *partner symmetry* for the original symmetry.

Both symmetries are related by a nonlocal recursion relation so that at least one of them is a nonlocal symmetry.

Sheftel and Malykh [2] have demonstrated how to use partner symmetries for obtaining noninvariant solutions of heavenly equations of Plebañski that govern heavenly gravitational metrics. Also, they presented a classification of scalar second order partial differential equations (PDEs) with four variables that possess partner symmetries and contain only second derivative of the unknown [1]. The general form of a second-order PDE with four independent variables x, y, z, t that possesses partner symmetries and contains only second derivatives of the unknown u reads

$$\begin{aligned} F = {} & a_1(u_{ty}u_{xz} - u_{tz}u_{xy}) + a_2(u_{tx}u_{ty} - u_{tt}u_{xy}) + a_3(u_{ty}u_{xx} - u_{tx}u_{xy}) \\ & + a_4(u_{tx}u_{tz} - u_{tt}u_{xz}) + a_5(u_{tz}u_{xx} - u_{tx}u_{xz}) + a_6(u_{tt}u_{xx} - u_{tx}^2) \\ & + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_0 = 0, \end{aligned} \tag{1.1}$$

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where a_i and b_i are arbitrary constants. Partner symmetries, that make it possible to obtain noninvariant solutions of PDEs of the form (1.1) are generated by the recursion relation:

$$\begin{aligned}\tilde{\varphi}_t &= -\big(a_2u_{ty}+a_4u_{tz}-a_6u_{tx}+b_6-\omega_0\big)\varphi_t-\big(a_3u_{ty}+a_5u_{tz}+a_6u_{tt}+b_7\big)\varphi_x\\&\quad+\big(a_1u_{tz}+a_2u_{tt}+a_3u_{tx}-b_1\big)\varphi_y+\big(-a_1u_{ty}+a_4u_{tt}+a_5u_{tx}-b_3\big)\varphi_z,\\\tilde{\varphi}_x &= -\big(a_2u_{xy}+a_4u_{xz}-a_6u_{xx}-b_5\big)\varphi_t-\big(a_3u_{xy}+a_5u_{xz}+a_6u_{tx}-b_6-\omega_0\big)\varphi_x\\&\quad+\big(a_1u_{xz}+a_2u_{tx}+a_3u_{xx}+b_2\big)\varphi_y+\big(-a_1u_{xy}+a_4u_{tx}+a_5u_{xx}+b_4\big)\varphi_z,\end{aligned}\tag{1.2}$$

where φ and $\tilde{\varphi}$ are symmetry characteristics [3] and ω_0 is a constant. In (1.1) and (1.2) subscripts denote partial derivatives of u, e.g $u_{tx}=\partial^2u/\partial t\partial x$, $u_{xx}=\partial^2u/\partial x^2,\ldots$ The transformation (1.2) maps any symmetry φ of Eq. (1.1) to its partner symmetry $\tilde{\varphi}$.

In this paper, we study bi-Hamiltonian structure of PDEs of the general form (1.1). Some particular cases of Eq. (1.1) yield bi-Hamiltonian systems in $(3+1)$ dimensions which had been studied in the last decade [4–7]. In order to discuss its Hamiltonian structure we shall single out an independent variable t in (1.1) to play the role of 'time' and express the general equation as a pair of first-order nonlinear evaluation equations. Before doing this, to avoid complications we will use the following short-hand notation:

$$\begin{aligned}c_1 &= a_3u_{xy}+a_5u_{xz}+a_6v_x-b_6-\omega_0\\c_2 &= a_1u_{xz}+a_2v_x+a_3u_{xx}+b_2\\c_3 &= -a_1u_{xy}+a_4v_x+a_5u_{xx}+b_4\\c_4 &= a_2u_{xy}+a_4u_{xz}-a_6u_{xx}-b_5\\c_5 &= a_3v_y+a_5v_z+a_6Q+b_7\\c_6 &= a_1v_z+a_2Q+a_3v_x-b_1\\c_7 &= -a_1v_y+a_4Q+a_5v_x-b_3\\c_8 &= a_2v_y+a_4v_z-a_6v_x+b_0-\omega_0\\c_9 &= a_6v_{xx}-a_2v_{xy}-a_4v_{xz}\\c_{10} &= b_1u_{xy}+b_3u_{xz}+b_7u_{xx}+b_0\end{aligned}\tag{1.3}$$

where Q appearing in c_5, c_6 and c_7 is given in (1.4). We introduce $u_t=v$ as a second unknown and Eq. (1.1), with the use of (1.3), can be written in the two-component form as follow:

$$u_t=v,\qquad v_t=\frac{1}{c_4}[(b_6-\omega_0-c_1)v_x+c_2v_y+c_3v_z+c_{10}]\equiv Q.\tag{1.4}$$

From now on, in all calculations we will use the short-hand notation (1.3).

In Section 2, we present the first Hamiltonian structure of this system of equations. We start with a degenerate Lagrangian and construct its Dirac bracket [8] to find a Hamiltonian operator.

In Section 3, we construct a recursion operator in a matrix form using results of [1]. Recursion operator and the operator of the symmetry condition form a Lax pair for the two-component system.

In Section 4, we give explicitly the second Hamiltonian structure which shows that Eq. (1.4) is an integrable bi-Hamiltonian system under some constraints on the constant coefficients.

In Section 5, we show that under a suitable choice of constant coefficients a_i and b_i, the general system (1.1) is reduced to known bi-Hamiltonian systems given in [4–7,9].

In Section 6, we prove that Hamiltonian operators are compatible and satisfy Jacobi identity by using Olver's method [3].

2. Lagrangian and first Hamiltonian structure

There is a systematic way to derive the first Hamiltonian structure of (1.4). This method is used for Plebański's heavenly equations [4], Husain and mixed heavenly [6] equations, complex Monge–Ampère [5] and asymmetric heavenly equations [7]. We shall now apply it to the evolution system (1.4).

We start with the degenerate Lagrangian density for (1.4) given by

$$\begin{aligned}L &= \left(vu_t-\frac{v^2}{2}\right)(a_6u_{xx}-a_2u_{xy}-a_4u_{xz}-b_5)-\frac{u_t}{3}(a_1u_z+a_3u_x)u_{xy}\\&\quad-\frac{u_t}{3}(a_5u_x-a_1u_y)u_{xz}+\frac{u_t}{3}(a_3u_y+a_5u_z)u_{xx}\\&\quad+\frac{u_t}{2}(b_2u_y+b_4u_z+2b_6u_x)+\frac{1}{2}(b_7u_x+b_1u_y+b_3u_z)u_x-b_0u.\end{aligned}\tag{2.1}$$