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On embeddings of almost complex manifolds in almost complex Euclidean spaces

Antonio J. Di Scala[a,*], Naohiko Kasuya[b], Daniele Zuddas[c]

[a] Dipartimento di Scienze Matematiche 'G.L. Lagrange', Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Torino, Italy

[b] Department of Social Informatics, Aoyama Gakuin University, 5-10-1 Fuchinobe, Chuo-ku, Sagamihara, Kanagawa 252-5258, Japan

[c] Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

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ABSTRACT

We prove that any compact almost complex manifold (M^{2m}, J) of real dimension $2m$ admits a pseudo-holomorphic embedding in $(\mathbb{R}^{4m+2}, \tilde{J})$ for a suitable positive almost complex structure $\tilde{J}$. Moreover, we give a necessary and sufficient condition, expressed in terms of the Segre class $s_m(M, J)$, for the existence of an embedding or an immersion in $(\mathbb{R}^{4m}, \tilde{J})$. We also discuss the pseudo-holomorphic embeddings of an almost complex 4-manifold in $(\mathbb{R}^6, \tilde{J})$.

1. Introduction

In this article we give some existence results of pseudo-holomorphic embeddings of almost complex manifolds into almost complex Euclidean spaces. More precisely, we prove that such an embedding exists if the dimension of the ambient Euclidean space is at least $4m+2$, where $2m$ is the real dimension of the source manifold. We also give results about pseudo-holomorphic immersions and embeddings into $\mathbb{R}^{4m}$ under certain assumptions on the Chern class. The Euclidean space is endowed with a suitable non-standard almost complex structure, which is not integrable in general.

As a further result, we provide a condition for the existence of codimension-two pseudo-holomorphic embeddings of almost-complex 4-manifolds.

We notice that Theorem 1 represents a major improvement of the main result of [1], where the pseudo-holomorphic embedding was in $\mathbb{R}^{6m}$. We reduce the dimension of the ambient Euclidean space to $4m + 2$. It is the best possible result, since there is an obstruction for pseudo-holomorphic embeddings in $\mathbb{R}^{4m}$, as stated in Theorem 3. We also fix a mistake in the proof of the main result of [1] that has to do with the homotopy type of the space of linear complex structures on $\mathbb{R}^{2n}$. These spaces have been considered to be $(n-1)$-connected in [1], but this is false. In the present paper we prove a stronger result, which is represented by Theorem 1.

* Corresponding author.
E-mail addresses: antonio.discala@polito.it (A.J. Di Scala), nkasuya@si.aoyama.ac.jp (N. Kasuya), zuddas@kias.re.kr (D. Zuddas).

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The space of positive linear complex structures on $\mathbb{R}^{2n}$ is homotopy equivalent to the symmetric space $\Gamma(n) = \mathrm{SO}(2n)/\mathrm{U}(n)$. If $k \leq 2n-2$, the homotopy groups $\pi_k(\Gamma(n))$ are said to be stable. The stable homotopy groups of $\Gamma(n)$ have been computed by Bott [2], who showed that for $k \leq 2n-2$,

$$\pi_k(\Gamma(n)) \cong \pi_{k+1}(\mathrm{SO}(2n)) \cong \begin{cases} 0 & \text{for } k \equiv 1, 3, 4, 5 \\ \mathbb{Z} & \text{for } k \equiv 2, 6 \\ \mathbb{Z}_2 & \text{for } k \equiv 0, 7. \end{cases} \qquad (\mathrm{mod}\ 8)$$

For the unstable homotopy groups of $\Gamma(n)$, see [3–6].

Now we state our main results, which are the following.

Theorem 1. *Any almost complex manifold (M^{2m}, J) of real dimension $2m$ can be pseudo-holomorphically embedded in $(\mathbb{R}^{4m+2}, \tilde{J})$ for a suitable positive almost complex structure $\tilde{J}$.*

Notice that the codimension $2m+2$ improves the $4m$ of [1, Theorem 1] for $m > 1$.

We denote by $c(M, J)$ the total Chern class of (M, J), and by $s(M, J) = c(M, J)^{-1}$ the total Segre class of (M, J). Let $s_k(M, J) \in H^{2k}(M)$ be the $2k$-dimensional term of $s(M, J)$. We also define

$$I(M, J) = -\frac{1}{2}\langle s_m(M, J), [M]\rangle \in \mathbb{Z}.$$

Remark 2. $I(M, J)$ is an integer because the normal Euler number of an immersion of M^{2m} in $\mathbb{R}^{4m}$ is even by a theorem of Whitney [7]. Indeed, $-2I(M, J)$ is going to be the normal Euler number, see the proof of Theorem 3.

Theorem 3. *An almost complex manifold (M^{2m}, J) of real dimension $2m$ can be pseudo-holomorphically immersed in $(\mathbb{R}^{4m}, \tilde{J})$ for a suitable positive almost complex structure $\tilde{J}$ if and only if $I(M, J) \geqslant 0$. In this case, there is a self-transverse pseudo-holomorphic immersion $(M, J) \looparrowright (\mathbb{R}^{4m}, \tilde{J})$ with exactly $I(M, J)$ double points. Thus, (M, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^{4m}, \tilde{J})$, for some $\tilde{J}$, if and only if $I(M, J) = 0$.*

The first part of the following corollary is immediate because $s_1(S, J) = -c_1(S, J)$, hence $I(S, J) = 1 - g(S)$ for a closed Riemann surface (S, J), see also [8]. The second part is a consequence of known facts, that is the existence of a torus fibration on S^4, which is due to Matsumoto [9]. However, we incorporate this result in the corollary because this construction enlightens nicely and more concretely our results in the case of elliptic curves as a family of pseudo-holomorphic curves in $\mathbb{R}^4$.

Corollary 4. *(S, J) can be pseudo-holomorphically immersed in $(\mathbb{R}^4, \tilde{J})$, for some $\tilde{J}$, if and only if S is either a torus or a sphere. Moreover, $\tilde{J}$ can be suitably chosen so that $(\mathbb{R}^4, \tilde{J})$ admits a two-dimensional holomorphic singular foliation with the property that the regular leaves but one are pseudo-holomorphically embedded tori, the other regular leaf is a pseudo-holomorphically embedded cylinder $S^1 \times \mathbb{R}$, and the singular leaf is a pseudo-holomorphically immersed sphere with one node.*

For a closed, oriented 4-manifold M, we denote the signature of M by $\sigma(M)$.

Theorem 5. *Suppose that (M, J) is a closed almost complex 4-manifold such that $H^2(M)$ has no 2-torsion. Then, (M, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^6, \tilde{J})$, for some $\tilde{J}$, if and only if $\sigma(M) = \chi(M) = 0$ and $c_1(M, J) = 0$.*

Remark 6. Our proof of Theorem 1 also shows the existence of an isometric and pseudo-holomorphic embedding of an almost Hermitian manifold (M^{2m}, J, g) into $(\mathbb{R}^{2q_m}, \tilde{J}, g_0)$, where g_0 is the flat standard metric, $\tilde{J}$ is a suitable almost complex structure compatible with g_0 and q_m is a sufficiently large integer. Indeed, the proof of Theorem 1 starts with the use of the well-known theorem of Whitney to embed M into $\mathbb{R}^{4m+2}$. If we start our proof of Theorem 1 with an isometric embedding f of (M, g) into $(\mathbb{R}^{2q_m}, g_0)$ where q_m is determined as in Nash's theorem [10], then our proof shows the existence of the suitable $\tilde{J}$ compatible with g_0 such that f is also pseudo-holomorphic (for recent improvements of q_m see [11] and the references therein).

The paper is organized as follows. In Section 2 we address some preliminaries and fix notations. In Section 3 we prove Theorems 1 and 3. The proofs make use of Proposition 11, which is proved therein. In Section 4 we quickly recall basic facts about Lefschetz fibrations, and prove the second part of Corollary 4. Section 5 addresses the four-dimensional case, with the proof of Theorem 5. We conclude with some remarks in Section 6, providing also the sketch of an alternative proof of Theorem 5.

2. Preliminaries

Throughout this paper, manifolds are assumed to be connected, oriented, and smooth, that is of class C^∞. Maps between manifolds are also assumed to be smooth, if not differently stated.