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Variational contact symmetries of constrained Lagrangians

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ABSTRACT

The investigation of contact symmetries of re-parametrization invariant Lagrangians of finite degrees of freedom and quadratic in the velocities is presented. The main concern of the paper is those symmetry generators which depend linearly in the velocities. A natural extension of the symmetry generator along the lapse function $N(t)$, with the appropriate extension of the dependence in $\dot{N}(t)$ of the gauge function, is assumed; this action yields new results. The central finding is that the integrals of motion are either linear or quadratic in velocities and are generated, respectively by the conformal Killing vector fields and the conformal Killing tensors of the configuration space metric deduced from the kinetic part of the Lagrangian (with appropriate conformal factors). The freedom of re-parametrization allows one to appropriately scale $N(t)$, so that the potential becomes constant; in this case the integrals of motion can be constructed from the Killing fields and Killing tensors of the scaled metric. A rather interesting result is the non-necessity of the gauge function in Noether's theorem due to the presence of the Hamiltonian constraint.

1. Introduction

The method of group invariant transformations is an important tool for the study of differential equations. The generator of an invariant transformation for a differential equation is called Lie symmetry. The importance of Lie symmetries lies in the fact that they can be used in order to reduce the order of an ordinary differential equation or to assist in the solution of differential systems by reducing the number of independent variables (for partial differential equations) or the order of the equation (for ordinary differential equations). The Lie symmetries which leave invariant a variational integral are called Noether symmetries. Noether symmetries form a subalgebra of the Lie symmetries of the equations following from the variational integral. The characteristic of Noether symmetries is that to each symmetry there corresponds a (on mass shell) divergence free current along with the corresponding charge. The well known conservation laws of energy, angular momentum all follow from Noether symmetries. It is well known that conservation laws are important tools which can be used for the determination of integrable manifolds of a dynamical system in mathematical physics, biology, economics and other areas, for instance see [1–4].

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The determination of the Lie and Noether symmetries of a given differential equation consists of two steps, (a) the derivation of the symmetry conditions and (b) the solution of these conditions. The first step is formal and it is outlined in e.g. [5–7]. The second step can be a difficult task since the symmetry conditions can be quite involved. Tsamparlis & Paliathanasis in [8,9] proposed a geometric method for the solution of the symmetry conditions for regular Hamiltonian systems which describe the motion of a particle in a Riemannian space under the action of a potential. It was shown that the Lie and Noether symmetries of that system are related with the special projective algebra of the underlying space. This geometric approach has been extended to the determination of the Lie and Noether point symmetries of some families of partial differential equations [10,11].

A similar analysis has been established by Christodoulakis, Dimakis & Terzis [12] for constraint Lagrangians quadratic in the velocities, where it was shown that the point symmetries of equations of motion are exactly the variational symmetries (containing the time reparametrization symmetry) plus the scaling symmetry. The variational symmetries were seen to be the simultaneous conformal Killing vector fields of both the metric, defined by the kinetic term, and the potential.

These results have been applied in various areas, i.e. in classical mechanics, in general relativity and in cosmology; in each case new dynamical systems which admit symmetries were found for instance see [13–15,9,16–20] and references therein.

In this work we extend this geometric approach in order to study the Lie Bäcklund variational symmetries (or dynamical Noether symmetries) of constrained Lagrangian systems. Specifically, we study the symmetries which arise from contact transformations, the contact Noether symmetries. Contrary to the point transformations which are defined in the configuration space manifold, where the dynamical equations are defined, the contact transformations are defined in the tangent bundle thus they depend also in the velocities [21,22]. Such contact Noether symmetries of regular Lagrangian systems with a potential have been studied in [23] and it was shown that the corresponding conservation laws follow from those (second rank) Killing tensors of the kinetic metric whose contraction with the gradient of the potential is the gradient of a (gauge) function. Some important conservation laws of this kind in physics are: the Runge–Lenz vector field of the Kepler problem, the Ray–Reid invariant, and the Carter constant in Kerr spacetime [24–27]. As it will be shown, in the case of constrained Lagrangian systems the results are different from those of [23] in that the Noether contact symmetries are related to the Conformal Killing tensors of the underlying space. The method we develop can also be used for regular Lagrangian systems. It is furthermore interesting to note that, in the presence of constrains, the gauge function of Noether's theorem does not play a significant role: the presence of the quadratic constraint renders it non-essential in finding the conservation laws. The plan of the paper is as follows.

In Section 2, we give the basic theory of the Lie Bäcklund symmetries. In Section 3, we study the linear contact symmetries of constraint Lagrangian systems and we prove the main theorem of this work. The conservation laws which follow from the contact symmetries are studied in 4. In Section 5 we demonstrate the main results by two applications; (a) the determination of the variational contact symmetries for the geodesic equations of a B*iv* pp-wave spacetime and (b) the derivation of the Ray–Reid invariant of the corresponding constraint Hamiltonian system. Finally, in Section 6 we discuss our results.

2. Preliminaries

For the convenience of the reader, in this section we present the basic properties and definitions concerning the generalized symmetries.

Let $H = H\left(x^i, u^A, u^A_{,i}, u^A_{,ij}...\right)$ be a function which is defined in the infinite dimensional space $A_M = \left\{x^i, u^A, u^A_{,i}, u^A_{,ij}, \dots\right\}$. A set of differential equations can be expressed as a function $H = 0$ on this space. In this case, x^i and u^A are to be regarded as the n independent and m dependent variables respectively, while the remaining set of variables $u^A_{,i}, u^A_{,ij}...$ can be used to represent the derivatives of the dependent variables u^A with respect to the x^i. Hence, for $n = 1$ we can consider ordinary differential equations and for $n > 1$ a set of partial differential equations.

We shall say that a function H is invariant under the action of the following infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i \left(x^i, u^B, u^B_{,i}, u^B_{,ij}...\right) \tag{2.1a}$$

$$\bar{u}^A = u^A + \varepsilon \eta^A \left(x^i, u^B, u^B_{,i}, u^B_{,ij}...\right) \tag{2.1b}$$

$$\bar{u}^A_{,i} = u^A_{,i} + \varepsilon \eta^A_i \left(x^i, u^B, u^B_{,i}, u^B_{,ij}...\right) \tag{2.1c}$$

$$\dots$$

if there exists a function $\lambda\left(x^i, u, u_{,i}, u_{,ij}...\right)$ such that the following condition holds [5]

$$\mathcal{L}_X H = \lambda H, \tag{2.2}$$

with

$$X = \xi^i \left(x, [u]\right) \partial_i + \eta^A \left(x, [u]\right) \partial_{u^A} + \eta^A_i \left(x, [u]\right) \partial_{u^A_{,i}} + \cdots \tag{2.3}$$

being the generator of the infinitesimal transformation (2.1), in which the set of variables $u^A, u^A_{,i}, u^A_{,ij}, \dots$ is denoted by $[u]$.

When $H = 0$ is considered to be a set of differential equations, we need to implement additional conditions so that such a transformation maps solutions onto solutions. For example, if there exists a smooth enough function $h^A(x) = u^A$ which