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Approximate damped oscillatory solutions and error estimates for the perturbed Klein–Gordon equation ☆

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ABSTRACT

The influence of perturbation on traveling wave solutions of the perturbed Klein–Gordon equation is studied by applying the bifurcation method and qualitative theory of dynamical systems. All possible approximate damped oscillatory solutions for this equation are obtained by using undetermined coefficient method. Error estimates indicate that the approximate solutions are meaningful. The results of numerical simulations also establish our analysis.

1. Introduction

The Klein–Gordon equation is a very important equation in the field of physics, especially in quantum mechanics [1–10]. It represents the equation of motion of a quantum scalar or a pseudo-scalar field, which is a field whose quanta are spinless particles. The Klein–Gordon equation with cubic nonlinearity reads in the form

$$u_{tt} - k^2 u_{xx} + au - bu^3 = 0, \tag{1}$$

where a, b and k are nonzero real constants. Eq. (1) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a "splay wave" along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc. [1]. There have been many studies on traveling wave solutions of Eq. (1) [7–10]. The solutions obtained are of interest in nuclear physics, solid state physics, high-energy physics, nonlinear optics and many more.

Recently, the study of perturbed nonlinear equations has attracted much attention. Zhang studied the perturbed Klein–Gordon equation with quadratic nonlinearity in the (1 + 1)-dimension without local inductance and dissipation effect, and obtained exact traveling wave solutions by employing the auxiliary ordinary differential equation [11]. Z.Y. Zhang et al. also considered the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity. They then constructed some new exact solutions for the equation by using various different methods, such as the modified mapping method and the extended mapping method [12], the modified trigonometric function series method [13], the modified (G'/G)-expansion method [14], Jacobi elliptic function expansion method [15] and the dynamical system approach [16,17]. Sassaman and Biswas [18] investigated the perturbed phi-four equation

$$u_{tt} - k^2 u_{xx} - u + u^3 = \epsilon R, \tag{2}$$

where $R = \alpha u + \beta u_t + \gamma u_x + \delta u_{xt} + \lambda u_{tt} + \sigma u_{xxt} + \upsilon u_{xxxx}$, R represents the perturbation terms and ϵ is the perturbation coefficient. They gave the adiabatic variation of the soliton velocity of Eq. (2).

Because the perturbation inevitably arises in the context of long Josephson junction, it is meaningful to study the following perturbed Klein–Gordon equation

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$$u_{tt} - k^2 u_{xx} + au - bu^3 = \epsilon(\alpha u_t + \beta u_x), \tag{3}$$

where $\epsilon > 0$ shows that the system has perturbation. The perturbation term due to α accounts for dissipative losses in Josephson junction theory due to tunneling of normal electrons across the dielectric barrier, while the perturbation term due to β is generated by a small inhomogenous part of the local inductance [18].

In this paper, we study the perturbation effect on traveling wave solutions of Eq. (3). This paper is organized as follows. In Section 2, we carry out qualitative analysis for the dynamic system corresponding to Eq. (3). Existence of bell solitary wave solution, kink solitary wave solutions and oscillatory traveling wave solutions of Eq. (3) are given. In Section 3, we study the effect of perturbation on the traveling wave solutions of Eq. (3). In Section 4, approximate damped oscillatory solutions are obtained by using undetermined coefficient method. In Section 5, we give error estimates for the approximate damped oscillatory solutions gotten in Section 4. Finally, we give some numerical simulations for special cases of Eq. (3) to support our analytical results of oscillatory traveling wave solutions obtained in the above sections. Through the investigation in this paper, we can comprehend the influence of perturbation term on traveling wave solutions of Eq. (3).

2. Existence of bell solitary wave solution, kink solitary wave solutions and oscillatory traveling wave solutions of Eq. (3)

Assume that Eq. (3) has traveling wave solution in the form

$$u(x,t) = u(\xi), \quad \xi = x - \lambda t, \tag{4}$$

where λ is the propagation speed of a wave. Substituting (4) into Eq. (3) yields

$$u''(\xi) + \frac{(\lambda\alpha - \beta)\epsilon}{\lambda^2 - k^2} u'(\xi) + \frac{a}{\lambda^2 - k^2} u(\xi) - \frac{b}{\lambda^2 - k^2} u^3(\xi) = 0, \quad (\lambda^2 \neq k^2). \tag{5}$$

Thus, to study the existence of bounded traveling wave solutions of Eq. (3) is equivalent to study the existence of bounded solutions of Eq. (5).

Let $u'(\xi) = y$, then Eq. (5) reduces to the following planar dynamic system

$$\begin{cases} \frac{du}{d\xi} = y \triangleq P(u,y), \\ \frac{dy}{d\xi} = -\frac{(\lambda\alpha-\beta)\epsilon}{\lambda^2-k^2} y - \frac{a}{\lambda^2-k^2} u + \frac{b}{\lambda^2-k^2} u^3 \triangleq Q(u,y). \end{cases} \tag{6}$$

Owing to $\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial y} = -\frac{(\lambda\alpha-\beta)\epsilon}{\lambda^2-k^2}$, by Bendixson–Dulac's criterion [19], we have the following proposition for system (6).

Proposition 1. *If $(\lambda\alpha - \beta)\epsilon \neq 0$, then system (6) does not have any closed orbit or singular closed orbit with finite number of singular points on (u,y) phase plane. Further, there exists no periodic traveling wave solution or bell solitary wave solution of Eq. (3) as $(\lambda\alpha - \beta)\epsilon \neq 0$.*

In the (u,y) plane, the number of bounded singularities in system (6) depends on the number of solutions in equation of

$$f(u) = -\frac{a}{\lambda^2 - k^2} u + \frac{b}{\lambda^2 - k^2} u^3 = 0. \tag{7}$$

For the sake of simplification, we only consider the case when $ab > 0$ and $(\lambda\alpha - \beta)(\lambda^2 - k^2) > 0$ throughout the paper. Other cases can be discussed similarly. Clearly, system (6) has three singular points under the condition $ab > 0$. Denote them by $O(0,0), P_-(-\sqrt{a/b}, 0)$ and $P_+(\sqrt{a/b}, 0)$.

Now, we study the types of singular points of (6) by the theory of planar dynamical systems [20,21].

2.1. Finite singular points of the system (6)

2.1.1. In the case of $\epsilon = 0$

In this case, system (6) is a Hamiltonian system with Hamiltonian function

$$H(u,y) = \frac{1}{2} y^2 + \frac{a}{2(\lambda^2 - k^2)} u^2 - \frac{b}{4(\lambda^2 - k^2)} u^4.$$

It is easy to obtain the types of singular points of (6) as follows:

(1) If $a(\lambda^2 - k^2) < 0$, then O is a saddle point; P_+ and P_- are centers.
(2) If $a(\lambda^2 - k^2) > 0$, then O is a center; P_+ and P_- are saddle points.

2.1.2. In the case of $\epsilon > 0$.

Set the Jacobi matrix of the linearized system of system (6) at singular points O and $P_\pm$ are

$$J(O) = \begin{pmatrix} 0 & 1 \\ -\frac{a}{\lambda^2-k^2} & -\frac{(\lambda\alpha-\beta)\epsilon}{\lambda^2-k^2} \end{pmatrix},$$

and

$$J(P_\pm) = \begin{pmatrix} 0 & 1 \\ \frac{2a}{\lambda^2-k^2} & -\frac{(\lambda\alpha-\beta)\epsilon}{\lambda^2-k^2} \end{pmatrix},$$

respectively. To relate conveniently, let

$$\Delta_0 = (\lambda\alpha - \beta)^2 \epsilon^2 - 4a(\lambda^2 - k^2),$$

$$\Delta = (\lambda\alpha - \beta)^2 \epsilon^2 + 8a(\lambda^2 - k^2).$$

(1) If $a(\lambda^2 - k^2) < 0$, then O is a saddle point, P_+ ,P_- are stable node points as $\Delta > 0$, and P_+ ,P_- are stable focus points as $\Delta < 0$.
(2) If $a(\lambda^2 - k^2) > 0$, then P_+ ,P_- are saddle points, O is a stable node point as $\Delta_0 > 0$, and O is a stable focus point as $\Delta_0 < 0$.

2.2. Infinite singular points of the system (6)

Applying Poincaré transformation to analyze singular points at infinity of system (6), it is clear that there exist a couple of singular points at infinity E_1 and E_2 on y axis, meanwhile, the circumference of Poincaré disk is orbits. We can also prove the following results.