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C^*-algebras of holonomy-diffeomorphisms & quantum gravity II

Johannes Aastrup *, Jesper Møller Grimstrup

Institut für Analysis, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

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ABSTRACT

We introduce the holonomy-diffeomorphism algebra, a C^*-algebra generated by flows of vector fields and the compactly supported smooth functions on a manifold. We show that the separable representations of the holonomy-diffeomorphism algebra are given by measurable connections, and that the unitary equivalence of the representations corresponds to measured gauge equivalence of the measurable connections. We compare the setup to Loop Quantum Gravity and show that the generalized connections found there are not contained in the spectrum of the holonomy-diffeomorphism algebra in dimensions higher than one.

This is the second paper of two, where the prequel gives an exposition of a framework of quantum gravity based on the holonomy-diffeomorphism algebra.

1. Introduction

Most fundamental theories of physics have connections among their basic variables, like the standard model of particle physics and the Ashtekar formulation of general relativity. It is therefore important, especially with respect to a quantization of these theories, to consider functions of connections, i.e. observables, within these theories. Probably the most well known example of such functions are the Wilson loops, i.e. traces of the holonomies of loops along closed paths; but also open paths have been considered, in particular when the observables have to act on fermions.

One problem we have encountered in our attempt [1] to merge elements of canonical quantum gravity with noncommutative geometry is that variables like the Wilson loops, and related variables, tend to discretize the underlying spaces. Therefore in this paper we will commence the study of an algebra of "functions" of smeared objects in order to avoid this discretization. More concretely we will study a C^*-algebra generated by flows of vector fields on a manifold M and the smooth compactly supported functions on M. Flows of vector fields constitute a natural notion of families of paths, and when evaluated on a connection in the spin-bundle S, naturally gives an operator on the spinors, i.e. on $L^2(M, S)$, and not like a path just acting on one point in M. The holonomy-diffeomorphism algebra is defined as the C^*-algebra generated by the flows and the smooth function with norm given by the supremum over all the smooth connections. In this setup, smooth connections are viewed as representations of the holonomy-diffeomorphism algebra. It is the holonomy-diffeomorphism algebra which is our candidate for an algebra of observables.

One test to see if a given algebra of observables is suitable is to look at the spectrum of the algebra, i.e. the space of irreducible representations modulo unitary equivalence. The main result in this paper is that all non-degenerate separable representations of the holonomy-diffeomorphism are given by so called measurable connections. These are objects, which

* Corresponding author.
E-mail addresses: aastrup@math.uni-hannover.de (J. Aastrup), jesper.grimstrup@gmail.com (J.M. Grimstrup).

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are similar to the generalized connections encountered in Loop Quantum Gravity (LQG), see [2], but which take the measure class of the Riemannian metrics into account instead of the measure class of the counting measure. The measurable connections of course contain the smooth connections.

This means that the generalized connections, which dominate the spectrum found in LQG, are excluded from the spectrum of the holonomy-diffeomorphism algebra. We believe this is a very significant result since several open problems in LQG can be traced back to the singular nature of the generalized connections: for instance the non-separable kinematic Hilbert space, the lack of weakly continuous operators and the ensuing inability to construct an off-shell constraint algebra (see [3]). It is therefore highly desirable to find a natural algebra of observables, which avoids the generalized connections and has a spectrum, which is as close as possible to the classical configurations space. The algebra of holonomy-diffeomorphisms meets this challenge and is furthermore, from a physical point of view, a natural choice of algebra since it simply encodes how "stuff" is moved around on a spatial manifold.

This paper is the second of two papers. Where its prequel [4] is concerned with an exposition of a mathematical framework of quantum gravity based on the holonomy-diffeomorphism algebra this paper is solely concerned with the mathematical analysis of this algebra.

The paper is organized as follows:

In Section 2 we define the holonomy-diffeomorphism algebra.

In Section 3 we define the flow algebra. This algebra is constructed as a quotient of the cross product of the group generated the flows of the vector fields and the compactly supported smooth functions on the manifold. The ideal in this cross product, which is divided out, is the relation of local reparametrization. In particular the representations defining the holonomy-diffeomorphism algebra also give representations of the flow algebra. We show that separable non degenerate representations of this flow algebra are given by so called measurable connections, and show the unitary equivalence between these measurable connection is given by measurable gauge equivalence.

In Section 4 we compare our setup with the LQG setup. The generalized connections appearing in LQG also give rise to representations of the flow algebra, however non-separable representations. We show that the generalized connections can be obtained as the representations of a discretized version of the flow algebra.

In Section 5 we study the properties of the representations of the holonomy-diffeomorphism algebra given by smooth connections. In particular we show that a connection is irreducible if and only if the corresponding representation of the holonomy-diffeomorphism algebra is irreducible, and give some structure of the separable part of the spectrum. In the second part of Section 5 we show that if the dimension of the manifold is bigger than 1 the representations defined in Section 4 coming from generalized connections are not contained in the spectrum of the holonomy-diffeomorphism algebra. We do, however, not know if there are other non-separable representations contained in the spectrum of the holonomy-diffeomorphism algebra.

2. The holonomy-diffeomorphism algebra

Let M be a connected manifold and S a vector bundle over M. We assume that S is equipped with a fiber wise metric. This metric ensures that we have a Hilbert space $L^2(M, \Omega^{\frac{1}{2}} \otimes S)$, where $\Omega^{\frac{1}{2}}$ denotes the bundle of half densities on M. Given a diffeomorphism $\phi : M \to M$, this acts unitarily on $L^2(M, \Omega^{\frac{1}{2}})$ via the pullback of forms. We denote the pullback of ϕ with ϕ^*.

Let X be a vector field on M, which can be exponentiated, and let ∇ be a connection on S. Denote by $t \to e^{tX}$ the corresponding flow. Given $m \in M$ let γ be the curve

$$\gamma(t) = e^{(1-t)X}(e^{1X}(m))$$

running from $e^{1X}(m)$ to m. We define the operator

$$e^X_\nabla : L^2(M, \Omega^{\frac{1}{2}} \otimes S) \to L^2(M, \Omega^{\frac{1}{2}} \otimes S)$$

in the following way:

Let $\xi \in L^2(M, \Omega^{\frac{1}{2}} \otimes S)$ be locally over $(e^{1X}(m))$ of the form $\sum_i \omega_i \otimes s_i$, where ω_i's are elements in $\Omega^{\frac{1}{2}}$ and s_i a section in S. The value of $(e^X_\nabla)(\xi)$ in the point m is given as

$$\sum_i ((e^{1X})^*(\omega_i)(m)) \otimes (\mathrm{Hol}(\gamma, \nabla) s_i(e^{1X}(m))),$$

where $\mathrm{Hol}(\gamma, \nabla)$ denotes the holonomy of ∇ along γ. If the connection ∇ is unitary with respect to the metric on S, then e^X_∇ is a unitary operator.

If we are given a system of unitary connections $\mathcal{A}$ we define an operator valued function over $\mathcal{A}$ via

$$\mathcal{A} \ni \nabla \to e^X_\nabla,$$

and denote this by e^X. Denote by $\mathcal{F}(\mathcal{A}, \mathcal{B}(L^2(M, \Omega^{\frac{1}{2}} \otimes S)))$ the bounded operator valued functions over $\mathcal{A}$. This forms a C^*-algebra with the norm

$$\|\Psi\| = \sup_{\nabla \in \mathcal{A}} \{\|\Psi(\nabla)\|\}, \quad \Psi \in \mathcal{F}(\mathcal{A}, \mathcal{B}(L^2(M, \Omega^{\frac{1}{2}} \otimes S))).$$