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On complete stationary vacuum initial data

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ABSTRACT

We describe a proof of M.T. Anderson's result (Anderson, 2000) on the rigidity of complete stationary initial data for the Einstein vacuum equations in spacetime dimension $3+1$, under an extra assumption on the norm of the stationary Killing vector field. The argument only involves basic comparison geometry along with some Bochner–Weitzenböck formula techniques. We also discuss on the possibility to extend these techniques to higher dimensions.

1. Introduction

In General Relativity, it is a natural task to try and classify spacetime solutions of the Einstein equations under geometric requirements. Many basic questions are still wide open, even in the case of the *vacuum Einstein equations* where the Ricci curvature tensor of the spacetime metric vanishes. However, significant progress has been done in particular cases, typically in presence of isometries. Among the simplest examples comes the study of *spherically symmetric*, Ricci-flat spacetimes in dimension $3+1$. The Birkhoff theorem asserts that such spacetimes are locally isometric to one of the maximally extended Schwarzschild spacetimes[1].

In this note, we restrict our attention to the class of spacetimes that are invariant under isometries in the time-direction. More precisely, we are interested here in spacetimes $(\mathcal{N}, \gamma)$, solutions of the Einstein equations, in the special case of vanishing energy–momentum tensor and cosmological constant (hence Ricci-flat), which admit a timelike Killing vector field ξ. In order to avoid pathologies, we moreover assume that the orbits of this vector field are diffeomorphic to $\mathbb{R}$, and that no closed timelike curves occur in the spacetime. Such spacetimes are called *stationary*; they are of considerable interest in General Relativity since they are expected to describe the final state of the gravitational collapse of a star into a black hole. We refer the interested reader to [2] for a survey on stationary spacetimes.

A simple, but fundamental class of such spacetimes is the class of *static* spacetimes. These are stationary spacetimes $(\mathcal{N}, \gamma)$ such that the orthogonal distribution with respect to the Killing vector field ξ is integrable. An equivalent formulation is to say that $(\mathcal{N}, \gamma)$ takes the form of a warped product

$$\mathbb{R} \times_u M := (\mathbb{R} \times M, -u^2 dt^2 + g),$$

where M is a spacelike hypersurface of $\mathcal{N}$ whose induced metric is the Riemannian metric g and u is a smooth, positive function on M. The fact that $\mathbb{R} \times_u M$ is a Ricci-flat spacetime is equivalent to the fact that the data (M, g, u) satisfies the following conditions:

$$\begin{aligned} \operatorname{Hess}_g u &= ug \\ \Delta_g u &= 0. \end{aligned} \tag{1}$$

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One also says that this static spacetime is *vacuum*, which refers to the fact that the energy–momentum tensor of general relativity is chosen to be zero.

The problem of classifying *static vacuum* spacetimes is therefore expressed as the problem of finding all positive solutions (g, u) of the above system. Fundamental examples of such static spacetimes are the Schwarzschild spacetimes. They have the expression

$$g = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 \sigma_{\mathbb{S}^{n-1}}, \qquad u = \left(1 - \frac{2m}{r^{n-2}}\right)^{1/2}$$

on the manifold $M = \left((2m)^{1/(n-2)}, +\infty\right) \times \mathbb{S}^{n-1}$, where $m \in \mathbb{R}$ is a parameter called the *mass*. Some rigidity statements hold in spacetime dimension $n + 1 = 4$. For instance, Bunting and Masood-Ul-Alam were able to prove that Schwarzschild spacetimes are the only static vacuum ones which have the further property to be asymptotically flat [3].

In the more general setting of *stationary vacuum* spacetimes, the classification in dimension $3 + 1$ of the asymptotically flat ones and the uniqueness of *Kerr* spacetimes has been a major problem of mathematical relativity for the last decades. We will not develop further on this question and refer the reader to [4] and references therein. In both cases, the spacetimes considered here may exhibit a black hole region and, as for the Schwarzschild and Kerr examples, may fail to be geodesically complete.

Instead, we focus here on stationary vacuum spacetimes which are moreover complete. The first rigidity result in this setting comes from Lichnerowicz [5], under the further assumptions that the spacetimes considered are $3 + 1$ dimensional and asymptotically flat. He obtains that only the Minkowski spacetime $\mathbb{R}^{3,1}$ fulfills these properties (see also Einstein and Pauli [6]).

Much more recently, Anderson [7] proved the corresponding result without the asymptotic flatness assumption.

Theorem 1.1 (*Anderson, 2000*)**.** *Let $(\mathcal{N}, \gamma)$ be a 4-dimensional complete stationary vacuum spacetime. Then $(\mathcal{N}, \gamma)$ is isometric to $(\mathbb{R} \times M, -dt^2 + g)$, for some flat complete Riemannian manifold (M, g).*

The proof of this result in [7] uses the full power of Cheeger–Fukaya–Gromov collapsing theory, with refinements specific to dimension three, which makes it far from elementary.

However, Case [8] (and subsequently Catino [9]) recently came back to the static vacuum setting and proved that all complete static vacuum $n + 1$-dimensional spacetimes $(\mathcal{N}, \gamma)$ take the form of a product $(\mathbb{R} \times M, -dt^2 + g)$, where (M, g) is a complete Ricci-flat n-dimensional Riemannian manifold. Their techniques are less sophisticated, relying on the Bochner formula, as well as comparison arguments à la Bakry–Émery.

In this paper, we will see how the same kind of techniques (and indeed without Bakry–Émery) can be adapted to provide a proof of rigidity in the stationary case, Theorem 2.2, in dimension $n+1 = 4$ and under a suitable completeness assumption (instead of requiring the space–time to be complete, we assume a natural metric on the orbit space is complete). The point is, even though our proof does not reach the full generality of Anderson's, it remains quite elementary. Note also that the stationary case is a bit more challenging than the static case, for the contribution of the non-trivial connection on the line bundle induces a contribution to the Ricci curvature which turns out to have a bad sign. This technicality is overcome by a conformal trick in dimension $3 + 1$. In higher dimension, one can derive similar formulas for stationary initial data but they are harder to control. We discuss them at the end of the paper.

2. The setting, in dimension $3 + 1$

Definitions of stationary spacetimes existing in literature can vary depending on the authors and the context, although all of them assume the existence of a timelike Killing vector field.[1] We adopt the following definition in our work (compare with [7] and [10, Chap. XIV]).

Definition 2.1. A $(n + 1)$-dimensional spacetime $(\mathcal{N}, \gamma)$ is called *stationary* if it has no closed timelike curves and if there exists a timelike Killing vector field ξ on $\mathcal{N}$ whose orbits are complete.

As mentioned in [11], the *chronological* assumption, corresponding to the non-existence of closed timelike curves, together with the orbit completeness prevent pathologies of the space of orbits.[2] In fact, a stationary spacetime $(\mathcal{N}, \gamma)$ in the sense of the above definition can be seen as a principal $\mathbb{R}$-bundle over the space of orbits M which is a smooth manifold diffeomorphic to any spacelike hypersurface of $\mathcal{N}$ (see Geroch [12]).

1 Note however that this is no longer exact in the context of asymptotically flat spacetimes with a black hole region, where the Killing vector field is usually asked to be timelike only in the asymptotic region, see *e.g.* [4]

2 Without this assumption, an example of pathological spacetime is the 2-dimensional torus equipped with the Minkowski metric $-dx^2 + dy^2$. The orbits of the timelike Killing vector field $\xi = \sqrt{2}\, \partial_x + \partial_y$ are diffeomorphic to $\mathbb{R}$, but the orbit space is not a smooth manifold.