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The symmetric elliptic and hyperbolic restricted 3-body problem on the unit circle

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ABSTRACT

We study the restricted 3-body problem with the constriction of motion to the unit circle. First, we study the 2-body problem on the unit circle and give the explicit solutions for a regularized version of the equations of motion for any initial data. We classify the motions in elliptic, parabolic, hyperbolic type and an equilibrium state. Then, we analyze the restricted 3-body problem on the unit circle when the primary bodies are performing elliptic and hyperbolic motions. We show the existence of just one equilibrium state when the masses of primary bodies are equal and we exhibit the hyperbolic structure of this equilibrium point via an exponential dichotomy. In the last part we regularize the equations of motion. We show the global dynamics and some periodic solutions with its respective period.

1. Introduction and statement of the main results

The n-body problem, the frame of this paper, consists of n point bodies such that their motion is governed by gravitational forces among the bodies. The equations of motion are given by

$$m_i\ddot{\mathbf{x}}_i = F_i(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{j=1, j\neq i}^{n} \frac{Gm_im_j(\mathbf{x}_j - \mathbf{x}_i)}{\|\mathbf{x}_j - \mathbf{x}_i\|^3}, \quad i = 1, 2, \dots, n, \tag{1}$$

where the position of the body with mass m_i is denoted by $\mathbf{x}_i$ ($\mathbf{x}_i \in \mathbb{R}^3$). The function $F_i : \mathbb{R}^{3n} \to \mathbb{R}^3$ stands for the total force applied over the body with mass m_i. G is the universal gravitational constant. The dot over $\mathbf{x}_i$ means $d\mathbf{x}_i/dt$. We assume that $\|\cdot\|$ is a suitable norm. It is a well-known fact that just for the case $n = 2$ the n-body problem is completely solved, corresponding the solutions to conics. For the case $n = 3$, the explicit solutions have been shown just for some initial data. These solutions correspond to the homographic solutions, that is solutions where the ratio of the mutual distances are constant during all motion.

A simple formulation of the 3-body problem was given by Euler in 1772. He stated for the first time a *restricted* version of the 3-body problem and although it has been studied for many top researchers, they have not been successful in the sense of

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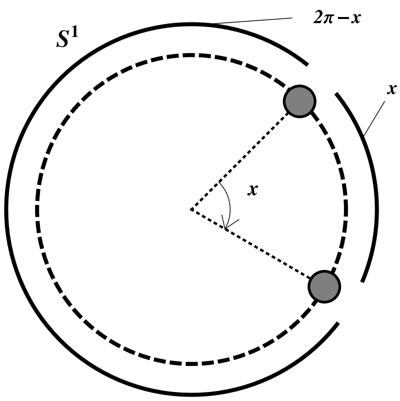

Fig. 1. The Kepler's problem on S^1.

giving the explicit solutions for any initial data. This restricted three body problem consists of three point bodies with motion on a fixed plane; two of them with mass m_1 and m_2 and the third with a negligible mass respect to the values m_1 and m_2. The former are called *primary bodies* and the latter is called either *secondary* or *infinitesimal body*. Thus the infinitesimal body does not affect the motion of the primary bodies, which are performing one solution of the 2-body problem, but these bodies determine the motion of the infinitesimal one. The target of the problem is to describe the dynamics of the infinitesimal body.

The subject of this paper is the restricted 3-body problem with the constriction of motion to the unit circle S^1, defining what we name *the restricted* 3-*body problem on the unit circle*.

We assume that the gravitational force is the negative of the gradient of a function $V : \mathbb{R}^{3n} \to \mathbb{R}$ called *potential*, namely $F = -\nabla V$. Thus, Eq. (1) can be rewritten as

$$m_i \ddot{\mathbf{x}}_i = -\nabla_{\mathbf{x}_i} V, \qquad V(\mathbf{x}_1, \ldots, \mathbf{x}_n) = -G \sum_{i<j} \frac{m_i m_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}. \tag{2}$$

From expression (2) we obtain the potential associated to the Kepler problem (the 2-body problem with one of the bodies fixed)

$$V(\mathbf{x}) = -\frac{\gamma}{\|\mathbf{x}\|}, \quad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2.$$

The constant γ depends on the masses and the universal gravitational constant. Adding the constriction of motion to S^1 in the plane (S^1 centered at the origin of $\mathbb{R}^2$) we obtain the potential for the Kepler problem on the unit circle

$$V(x) = -\frac{\gamma}{x} - \frac{\gamma}{2\pi - x}, \tag{3}$$

where x is the arc length measured from the attraction center to the body in clockwise sense as defined in [1] (see Fig. 1). Since the identification $S^1 \cong \mathbb{R}/2\pi$, the domain for the function (3) is $(0, 2\pi)$.

In some sense, the elliptic collinear restricted 3-body problem is the closest statement to our subject. In [2] it has been proved the no integrability of this problem when the masses of the primary bodies are very different. In [3] and [4] the authors tackled the same problem with symmetry in the masses, they show the global dynamics after doing several changes of variables and by introducing symbolic dynamics.

From [1], we know that the 2-body problem on S^1 has elliptic, parabolic, hyperbolic solutions and one equilibrium point. In this paper we give the global dynamics for the restricted 3-body problem on S^1 when the primary bodies performs elliptic and hyperbolic motions in the symmetric case, i.e. $m_1 = m_2 = 1/2$ with all binary collisions regularized.

The main results of this work are:

Theorem 1. *The elliptic and hyperbolic symmetric restricted* 3-*body problem on* S^1 *possess the following global dynamics (regularized).*

(i) *An equilibrium position with hyperbolic structure.*
(ii) *Elliptic symmetric motions.*
(iii) *Parabolic symmetric motions.*
(iv) *Hyperbolic symmetric motions.*

In order to obtain families of periodic orbits, we state three different types of orbits:

- *Collision–ejection* type: those orbits that present just binary collisions.
- *Big-bang* type: those orbits that present just triple collisions.
- *Mixed* type: those orbits that present binary and triple collisions alternately.