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The degree of mobility of Einstein metrics

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ABSTRACT

Two (pseudo-)Riemannian metrics are called projectively equivalent if their unparametrized geodesics coincide. The degree of mobility of a metric is the dimension of the space of metrics that are projectively equivalent to it. We give a complete list of possible values for the degree of mobility of Riemannian and Lorentzian Einstein metrics on simply connected manifolds, and describe all possible dimensions of the space of essential projective vector fields.

1. Introduction

The aim of this article is to study Einstein metrics (i.e., such that the Ricci curvature is proportional to the metric) of Riemannian and Lorentzian signature in the realm of projective geometry.

Recall that two (pseudo-)Riemannian metrics g and $\bar{g}$ on a manifold M are called *projectively equivalent*[1] if their unparametrized geodesics coincide. Clearly, any constant multiple of g is projectively equivalent to g. A generic metric does not admit other examples of projectively equivalent metrics, see [1]. If two metrics g, $\bar{g}$ are *affinely equivalent*, that is, if their Levi-Civita connections coincide, then they are also projectively equivalent. Affinely equivalent metrics are well-understood at least in Riemannian [2,3] and Lorentzian signature [4,5], see also Lemma 9. The case of arbitrary signature is much more complicated, see [4] or the more recent article [6] for a local description of all such metrics.

The theory of projectively equivalent metrics has a long and rich history—we refer to the introductions of [7,8] or to the survey [9] for more details, and focus on Einstein metrics in what follows.

Einstein metrics are very natural objects in projective geometry. For instance, as shown in [7], the property of a metric g to be Einstein is projectively invariant in the following sense: any metric that projectively equivalent and not affinely equivalent to an Einstein metric is also Einstein. A more educated point of view on the whole subject is the following: a projective geometry, given by a class of projectively equivalent connections (not necessarily Levi-Civita connections), is an example of a parabolic geometry, a special case of a Cartan geometry, see the monographs [10,11]. As shown in [12], the metrics with Levi-Civita connection contained in the given projective class are in one-to-one correspondence with the solutions of a certain overdetermined system of partial differential equations. This system is the so-called first Bernstein–Gelfand–Gelfand equation (see [13,14]) and, as shown in [15], Einstein metrics correspond to a special class of solutions called normal.

The *degree of mobility* $D(g)$ of a (pseudo-)Riemannian metric g is the dimension of the space of g-symmetric solutions of the PDE (2). As we explain in Section 2, nondegenerate solutions of (2) are in one-to-one correspondence with the metrics projectively equivalent to g. Hence, intuitively, $D(g)$ is the dimension of the space of metrics projectively equivalent to g.

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[1] The notions "geodesically equivalent" or "projectively related" are also common.

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0393-0440/

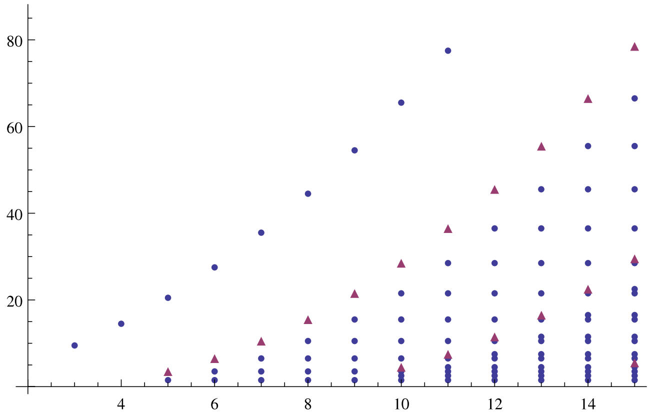

Fig. 1. Degree of mobility $D(g)$ from Theorem 1 for $3 \leq \dim M \leq 15$. The triangles denote the additional values for Lorentz signature.

We have $D(g) = 1$ for a generic metric g and $D(g) \geq 2$ if g admits a projectively equivalent metric that is nonproportional to g. As our main result, we determine all possible values for the degree of mobility $D(g)$ of Riemannian and Lorentzian Einstein metrics, locally or on simply connected[2] manifolds. Let us denote by "$[\alpha]$" the integer part of a real number α.

Theorem 1. *Let (M, g) be a simply connected Riemannian or Lorentzian Einstein manifold of dimension $n \geq 3$. Suppose g admits a projectively equivalent but not affinely equivalent metric.*

Then, the degree of mobility $D(g)$ is one of the numbers ≥ 2 from the following list:

- $\frac{k(k+1)}{2} + l$, *where* $n \geq 5$, $0 \leq k \leq n-4$ *and* $1 \leq l \leq [\frac{n+1-k}{5}]$ *for g Riemannian and Lorentzian.*
- $\frac{k(k+1)}{2} + l$, *where* $n \geq 5$, $k = n - 3 \bmod 5$, $2 \leq k \leq n-3$ *and* $l = [\frac{n+2-k}{5}]$ *for g Lorentzian.*
- $\frac{(n+1)(n+2)}{2}$.

Conversely, for $n \geq 3$ and each number $D \geq 2$ from this list, there exist simply connected n-dimensional Riemannian resp. Lorentzian Einstein manifolds admitting projectively equivalent but not affinely equivalent metrics and such that D is the degree of mobility $D(g)$.

In Theorem 1, the degree of mobility is at least 2 since we assumed that g admits a metric $\bar{g}$ projectively equivalent to g but not affinely equivalent to it. Suppose this assumption is dropped, that is, let us assume all metrics projectively equivalent to g are affinely equivalent to it. In this case the complete list of possible values of the degree of mobility of g can be easily obtained by combining Lemma 9 with methods similar to the ones used in Sections 3.2 and 3.4. It is

$$\{k(k+1)/2 + l : 0 \leq k \leq n-2,\ 1 \leq l \leq [(n-k)/2]\} \cup \{n(n+1)/2\}$$

if g is Einstein with nonzero scalar curvature and

$$\{k(k+1)/2 + l : 0 \leq k \leq n-4,\ 1 \leq l \leq [(n-k)/4]\} \cup \{n(n+1)/2\}$$

if g is Ricci flat.

It is well-known, see e.g. [16, p. 134], that if $D(g)$ is equal to its maximal value $(n+1)(n+2)/2$, then g has constant sectional curvature. Conversely, this value is attained on simply connected manifolds of constant sectional curvature. In view of this, the case $n = 3$ in Theorem 1 is trivial, since a 3-dimensional Einstein metric has constant sectional curvature and its degree of mobility takes the maximum value $D(g) = 10$.

For 4-dimensional Einstein metrics, we obtain the following statement as an immediate consequence of Theorem 1 (compare also Fig. 1):

Corollary 2. *Let (M, g) be a 4-dimensional Riemannian or Lorentzian Einstein manifold. Suppose $\bar{g}$ is projectively equivalent to g but not affinely equivalent. Then, g has constant sectional curvature.*

Corollary 2 was known before, see [7, Theorem 2] (or, alternatively, [17]), and it is actually true for metrics of arbitrary signature. However, our methods for proving Theorem 1 and Corollary 2 are different from that used in [17,7] (although we will rely on some statements from [7]). A special case of Corollary 2 was also considered in [5] where it was proven that 4-dimensional Ricci flat nonflat metrics cannot be projectively equivalent unless they are affinely equivalent. This result was generalized to Einstein metrics of arbitrary scalar curvature in [18]. Note that by [7, Theorem 1], the statement of Corollary 2 survives for arbitrary dimension under the assumption that both metrics are geodesically complete.

2 By definition, simple connectedness implies connectedness.