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Crossed modules of Hopf algebras and of associative algebras and two-dimensional holonomy

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ABSTRACT

After a thorough treatment of all algebraic structures involved, we address two dimensional holonomy operators with values in crossed modules of Hopf algebras and in crossed modules of associative algebras (called here crossed modules of bare algebras). In particular, we will consider two general formulations of the two-dimensional holonomy of a (fully primitive) Hopf 2-connection (exact and blur), the first being multiplicative the second being additive, proving that they coincide in a certain natural quotient (defining what we called the fuzzy holonomy of a fully primitive Hopf 2-connection).

1. Introduction

A crossed module of groups $\mathcal{X} = (\partial\colon E \to G, \triangleright)$ is given by a map $\partial\colon E \to G$ of groups, together with an action $\triangleright$ of G on E by automorphisms. This action must satisfy two natural properties (the Peiffer relations):

$$\partial(g \triangleright e) = g\, \partial(e)\, g^{-1}, \quad \text{for each } g \in G, \text{ and } e \in E \qquad \text{(first Peiffer relation)},$$

$$\partial(e) \triangleright f = e f e^{-1}, \quad \text{for each } e, f \in E \qquad \text{(second Peiffer relation)}.$$

Note that the second Peiffer law implies that $\ker(\partial)$ is an abelian subgroup of E. These are very flexible axioms. For example, given crossed modules $\mathcal{X} = (\partial\colon E \to G, \triangleright)$ and $\mathcal{X}' = (\partial\colon E' \to G', \triangleright)$, then $\mathcal{X} \times \mathcal{X}' = \big((\partial \times \partial')\colon E \times E' \to G \times G', \triangleright \times \triangleright'\big)$, with the obvious product action $\triangleright \times \triangleright'$, is a crossed module. On the other hand, if L is a subgroup of $\ker(\partial) \subset E$, such that $G \triangleright L \subset L$, which implies that L is normal (because of the second Peiffer relation), then $\mathcal{X}/L = (\partial\colon E/L \to G, \triangleright)$ with the obvious quotient action is also a crossed module. We note that morphisms of crossed module are defined in the obvious way.

Crossed modules of groups were invented by Whitehead in [1], naturally appearing in the context of homotopy theory, being algebraic models for homotopy 2-types [2] (for a modern account of this see [3,4]). Given a pointed fibration $F \to E \to B$, then the inclusion of the fiber F in the total space E induces a crossed module $\big(\pi_1(F) \to \pi_1(E)\big)$. A pointed pair of spaces (M, N) also has a fundamental crossed module $\big(\pi_2(M, N) \to \pi_1(N)\big)$, with the obvious boundary map and action of π_1 on π_2. The relation between group crossed modules and strict 2-groups (small categories whose sets of objects and of morphisms are groups, with all structure maps, including the composition, being group morphisms) was elucidated in [5],

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where we can find the first reference to the fact that the categories of crossed modules and of strict 2-groups are equivalent. This relation was fully generalized for crossed complexes and ω-groupoids in [6].

We can see a (strict) 2-group as being a small 2-category with a single object where all morphisms are invertible; [7]. Given a crossed module $\mathcal{X} = (\partial\colon E \to G, \triangleright)$, the 1-morphisms of the associated 2-group, denoted by $\mathcal{C}^\times(\mathcal{X})$, have the form $* \xrightarrow{g} *$, where $g \in G$, with the obvious composition. The 2-morphisms of $\mathcal{C}^\times(\mathcal{X})$ have the form below, composing vertically and horizontally (for conventions see 2.3.1):

$$\begin{tikzcd} * \arrow[r, bend left, "\partial(e)^{-1}g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\boxed{\times} \Uparrow e"] & * \end{tikzcd}, \quad \text{where } g \in G, \text{ and } e \in E.$$

A general theory of (Lie) 2-groups (including non-strict ones) appears in [7]. This theory parallels the theory of Lie-2-algebras (strict and non-strict), which was developed in [8]. We mention that a strict Lie 2-algebra is uniquely represented by a crossed module $\mathfrak{X} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ of Lie algebras (also called a differential crossed module), where $\triangleright$ is a left action of $\mathfrak{g}$ on $\mathfrak{e}$ by derivations, and $\partial\colon \mathfrak{e} \to \mathfrak{g}$ is a Lie algebra map, furthermore satisfying the (obvious) differential Peiffer relations. Given a crossed module $\mathcal{X} = (\partial\colon E \to G, \triangleright)$, of Lie groups, the induced Lie algebra map $\partial\colon \mathfrak{e} \to \mathfrak{g}$, and action by derivations of $\mathfrak{g}$ on $\mathfrak{e}$, defines a crossed module $\mathfrak{X} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$, of Lie algebras.

On the purely algebraic side, crossed modules of (Lie) groups and of Lie algebras arise in a variety of ways. Given a Lie group G, we have the actor crossed module of G, being $\mathcal{AUT}(G) = (G \xrightarrow{\mathrm{ad}} \mathrm{Aut}(G), \triangleright)$, where $\mathrm{Aut}(G)$ is the Lie group of automorphisms of G, acting in G in the obvious way and ad is the map that sends every $g \in G$ to the map $\mathrm{ad}_g\colon G \to G$ which is conjugation by g. If we have a central extension $\{1\} \to A \xrightarrow{i} B \xrightarrow{\partial} K \to \{1\}$ of groups, given any section $s\colon K \to B$ of $\partial\colon B \to K$, then $k \triangleright b \doteq s(k)\, b\, s(k)^{-1}$, where $k \in K$ and $b \in B$, is an action of K on B, by automorphisms independent of the chosen section, defining a crossed module with underlying group map being $\partial\colon B \to K$.

To a chain-complex $\mathcal{V} = (\dots \xrightarrow{\beta} V_n \xrightarrow{\beta} V_{n-1} \xrightarrow{\beta} \dots)$ of vector spaces we can associate crossed modules of Lie algebras and (in the finite dimensional case) of Lie groups, denoted respectively by $\mathfrak{GL}(\mathcal{V}) = \big(\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V}), \triangleright\big)$ and by $\mathrm{GL}(\mathcal{V})$; see [7,9]. The Lie algebra $\mathfrak{gl}_0(\mathcal{V})$ is made out of chain-maps $\mathcal{V} \to \mathcal{V}$, with the usual bracket. On the other hand, the Lie algebra $\mathfrak{gl}_1(\mathcal{V})$ is given by homotopies (degree one maps) $\mathcal{V} \to \mathcal{V}$, up to 2-fold homotopies. (The bracket is not the commutator of the underlying linear maps, the latter having degree two.) The 2-group associated to the crossed module $\mathrm{GL}(\mathcal{V})$ has a single object, has morphisms being the invertible chain maps $\mathcal{V} \to \mathcal{V}$ and 2-morphisms being the chain homotopies (up to 2-fold homotopy) between them. The vertical composition of 2-morphisms is induced by the sum of homotopies.

We can also consider pre-crossed modules of groups and of Lie algebras. These are defined similarly to crossed modules, however not imposing the second Peiffer condition. Crossed modules form a full subcategory of the category of pre-crossed modules (both in the group and Lie algebra cases). Moreover, any pre-crossed module can naturally be converted to a crossed module by dividing out the second Peiffer relations. This defines reflection functors {pre-crossed modules} $\to$ {crossed modules}.

Crossed modules of groups and of Lie algebra naturally have *free* objects. Let $\mathfrak{g}$ be a Lie algebra. Let also E be a set, with a map $\partial_0\colon E \to \mathfrak{g}$. We have a free differential crossed module $\mathfrak{F} = (\partial\colon \mathfrak{f} \to \mathfrak{g})$ on the map $\partial_0\colon E \to \mathfrak{g}$. Here $\partial\colon \mathfrak{f} \to \mathfrak{g}$ is a Lie algebra map, and we have a (set) inclusion $i\colon E \to \mathfrak{f}$, satisfying $\partial \circ i = \partial_0$. This free crossed module $\mathfrak{F}$ satisfies (and is defined by) the following universal property: if we have a differential crossed module $\mathfrak{X}' = (\partial'\colon \mathfrak{e}' \to \mathfrak{g}', \triangleright)$, a Lie algebra map $f\colon \mathfrak{g} \to \mathfrak{g}'$ and a set map $g_0\colon E \to \mathfrak{e}'$, satisfying $\partial' \circ g_0 = f \circ \partial_0$, then there exists a unique Lie algebra map $g\colon \mathfrak{f} \to \mathfrak{e}'$, extending g_0, and such that, furthermore, the pair $\big(g\colon \mathfrak{f} \to \mathfrak{e}', f\colon \mathfrak{g} \to \mathfrak{g}'\big)$ is a map of differential crossed modules $\mathfrak{F} \to \mathfrak{X}'$.

Models for the free differential crossed module on a set map $\partial_0\colon E \to \mathfrak{g}$ appear in [10,11]. Consider the action $\triangleright$ of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$ by left multiplication. Consider also the vector space $\bigoplus_{e\in E} \mathcal{U}(\mathfrak{g}).e$, with the obvious action $\triangleright$ of $\mathfrak{g}$. Consider the linear map $\partial_1\colon \bigoplus_{e\in E} \mathcal{U}(\mathfrak{g}).e \to \mathfrak{g}$ such that $\partial_1(a.e) = a \triangleright_{\mathrm{ad}} \partial_0(e)$ (here $a \in \mathcal{U}(\mathfrak{g})$ and $e \in E$), where $\triangleright_{\mathrm{ad}}$ is the action of $\mathcal{U}(\mathfrak{g})$ in $\mathfrak{g}$ induced by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$. Clearly $\partial_1(X \triangleright (a.e)) = [X, \partial_1(a.e)]$, where $X \in \mathfrak{g}$, $a \in \mathcal{U}(\mathfrak{g})$ and $e \in E$. Let $\mathfrak{f}_1$ be the free Lie algebra on the vector space $\bigoplus_{e\in E} \mathcal{U}(\mathfrak{g}).e$. We therefore have a Lie algebra map $\partial_2\colon \mathfrak{f}_1 \to \mathfrak{g}$, extending ∂_1. Since any linear map $V \to V$, V a vector space, induces a derivation at the level of the free Lie algebra on V, we have a Lie algebra action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{f}_1$, by derivations. Clearly this defines a pre-crossed module of Lie algebras. The free crossed module on the map $\partial_0\colon E \to \mathfrak{g}$ is obtained by converting the latter differential pre-crossed module into a crossed module of Lie algebras.

Crossed modules (of groups and of Lie algebras) are closely related to cohomology; see [12–14]. Namely, in the group case (not considering any topology), given a group K and a K-module A, there is a one-to-one correspondence between group cohomology classes $\omega \in H^3(K, A)$ and weak homotopy classes of crossed modules $(\partial\colon E \to G, \triangleright)$, fitting inside the exact sequence $\{0\} \to A \to E \xrightarrow{\partial} G \to K \cong G/\partial(E) \to \{1\}$. We actually have a group isomorphism considering Baer sums of crossed modules; [13].