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Strict deformation quantization of locally convex algebras and modules

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ABSTRACT

In this work various symbol spaces with values in a sequentially complete locally convex vector space are introduced and discussed. They are used to define vector-valued oscillatory integrals which allow to extend Rieffel's strict deformation quantization to the framework of sequentially complete locally convex algebras and modules with separately continuous products and module structures, making use of polynomially bounded actions of $\mathbb{R}^n$. Several well-known integral formulas for star products are shown to fit into this general setting, and a new class of examples involving compactly supported $\mathbb{R}^n$-actions on $\mathbb{R}^n$ is constructed.

1. Introduction

Deformation quantization as introduced in [1] comes in several different flavours: in formal deformation quantization one deforms the commutative pointwise product of the Poisson algebra of smooth functions on a Poisson manifold into a noncommutative *star product* as a formal associative deformation in the sense of Gerstenhaber [2] with deformation parameter $\hbar$. Here the general existence and classification for arbitrary Poisson manifolds is known and follows from Kontsevich's formality theorem [3], see [4] for an introductory textbook.

However, for many reasons formal deformations are not sufficient: for the original application to quantum mechanics one has to treat $\hbar$ as a positive number and not just as a formal parameter. But also applications beyond quantum theory require a more analytic framework. In particular, deformation quantization provides fundamental examples in noncommutative geometry where a C^*-algebraic formulation is needed.

In [5], Rieffel introduced a very general way to construct C^*-algebraic deformations based on a strongly continuous action of $\mathbb{R}^d$ on a C^*-algebra $\mathcal{A}$. For the smooth vectors $\mathcal{A}^\infty$ with respect to the action a product formula based on an oscillatory integral was established, generalizing the well-known Weyl quantization of $\mathbb{R}^{2n}$. In a second step, a matching C^*-norm is constructed, leading to a continuous field of C^*-algebras over the parameter space of $\hbar \in \mathbb{R}$. This construction and variants of it have by now found many applications in noncommutative geometry [6,5,7] and quantum physics, in particular in the context of quantum field theory on noncommutative spacetimes [8–12].

While for the construction of deformed C^*-algebras Rieffel's work is sufficient, it turns out that the first step of deforming the smooth vectors $\mathcal{A}^\infty$ is of interest already for its own sake: Rieffel worked with a Fréchet algebra with an isometric action.

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It is this situation which we want to generalize in various directions in the present work. First the restriction to a Fréchet algebra has to be overcome as there are several examples of interest which do not fall into this class. When interested in noncommutative spacetimes, a smooth structure in form of a deformation of the smooth functions is needed for many reasons. Thus we are interested in e.g. deformations of $\mathscr{C}_0^\infty(M)$. Moreover, together with a deformation of algebras one is interested in a possible deformation of modules as well. In the above example one might also be interested in a corresponding deformation of the distribution spaces $\mathscr{C}_0^\infty(M)'$. Hence we clearly have to pass beyond a Fréchet situation. Here we face several new phenomena: first of all products or module structures may be separately continuous without being continuous. In fact, there are many natural examples like this. Second, sequentially complete locally convex spaces need not be complete, with the distribution spaces as the most prominent examples. Third, the restriction to isometric actions, which is natural in the original C^*-setting, seems to be too restrictive in a more general locally convex framework. Again, many examples of interest show that one has to overcome this restriction.

As is well known, scalar-valued oscillatory integrals can be defined for more general functions than smooth functions with bounded derivatives—here the Hörmander symbols are a natural candidate. Thus we will adapt the notion of a symbol to the vector-valued case and study oscillatory integrals. These will be needed to handle actions of $\mathbb{R}^n$ which are not isometric but satisfy certain polynomial growth conditions. Compared to the scalar case the new feature is that for every continuous seminorm (of a defining system of seminorms) of the target space we have to allow for a specific growth. The examples show that we cannot expect to have a uniform growth for all seminorms.

The main result of this work is the construction of a Rieffel deformation for a sequentially complete locally convex algebra with a separately continuous product with respect to a smooth polynomially bounded action of $\mathbb{R}^n$ by automorphisms. Analogously, we give the corresponding deformation for a sequentially complete locally convex module with separately continuous module structure, provided the module structure is covariant for the $\mathbb{R}^n$-action. To this end we introduce the relevant symbol spaces and their oscillatory integrals based on a Riemann integral as we want to include sequentially complete spaces as well. This part is clearly of independent interest. We discuss several known examples within this framework and provide one new example of an action of $\mathbb{R}^n$ with compact support. A priori one can only guarantee exponential bounds for the derivatives of such an action, but by a particular construction we achieve polynomial growth behaviour. Actions of this type are needed in models of locally noncommutative spacetimes as introduced in [9,10]. In fact, the wish to have a smooth version of [10] was one of the main motivations to develop the above generalization of Rieffel's original work as a compactly supported action cannot be expected to be isometric for the seminorms of smooth functions. In the diploma thesis [13, Sect. 6.2] some aspects of the vector-valued oscillatory integrals were already anticipated.

It should be mentioned that there are still generalizations possible. One important step beyond Rieffel's original setting is to include actions of other Lie groups than $\mathbb{R}^n$. Here one first needs to find an analogue of Weyl quantization which then serves as universal deformation formula. This point of view was taken in the works of Bieliavsky et al., see e.g. [14–17]. While these works mainly deal with the C^*-algebraic deformation, in a more recent work [18], Bieliavsky and Gayral discuss also deformation aspects of Fréchet- and C^*-algebras based on symbol spaces and oscillatory integrals similar to ours. We leave it to a future investigation of whether their construction can be extended beyond the Fréchet case: in principle this looks very promising.

The paper is organized as follows. In Section 2 we introduce vector-valued symbols in the spirit of Hörmander symbols. However, the order as well as the type of the symbol may depend on the seminorm of the target space, a generalization needed to deal with the examples discussed later. We introduce in detail various symbol spaces, investigate the continuity properties of the usual algebraic manipulations, and show that the affine symmetries of the domain give continuous group actions on the symbol spaces. In particular, the translations act smoothly. In Section 3 we discuss the oscillatory integrals. Our approach follows the usual scalar case with the technical complication that we have to deal with many seminorms on the target instead of one. Thus a careful investigation of the polynomial growth is presented. The integrals are based on a Riemann integral for the smooth compactly supported functions as we want to include targets which are only sequentially complete. After this preparation, Section 4 is devoted to the deformation program. Based on the developed oscillatory integrals we extend Rieffel's construction to actions of $\mathbb{R}^n$ by automorphisms on sequentially complete locally convex algebras with separately continuous products and their modules. In case where the products are continuous also the resulting deformed products are continuous. For *-algebras, we also study positivity aspects of this deformation procedure. Extending results of [19], we show how to deform positive functionals on the original algebra to positive functionals on the deformed algebra.

Finally, Section 5 contains several examples of our general construction. First we discuss the usual action of $\mathbb{R}^{2n}$ on itself by translations and the induced action on various function spaces. Here in particular the scalar symbol spaces, the Schwartz space, and certain distribution spaces are discussed. This way we show that the well-known Weyl product formula, being defined pointwise for these spaces, can be understood as resulting from the oscillatory integral formulas. This is a nontrivial statement as in all cases the action is *not* isometric. As a second example, we consider in a Hilbert space setting unbounded operators which satisfy polynomial bounds with respect to the generators of a suitable action of $\mathbb{R}^n$. We then discuss deformations of their action on a suitable smooth subspace, thereby giving examples of the general module deformation which is typical for applications in quantum physics. The third example will be used in a future project for the construction of locally noncommutative spacetimes and corresponding quantum field theory models. It provides an action of $\mathbb{R}^n$ on $\mathbb{R}^n$ with compact support inside a given compact subset such that the induced action on the smooth functions is polynomially bounded. The difficulty is to pass from a trivially given exponential growth of the derivatives to a polynomial growth.