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Noncommutative complex structures on quantum homogeneous spaces

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ABSTRACT

A new framework for noncommutative complex geometry on quantum homogeneous spaces is introduced. The main ingredients used are covariant differential calculi and Takeuchi's categorical equivalence for quantum homogeneous spaces. A number of basic results are established, producing a simple set of necessary and sufficient conditions for noncommutative complex structures to exist. Throughout, the framework is applied to the quantum projective spaces endowed with the Heckenberger–Kolb calculus.

1. Introduction

Classical complex geometry is a subject of remarkable richness and beauty with deep connections to modern physics. Yet despite over twenty-five years of noncommutative geometry, the development of noncommutative complex geometry is still in its infancy. What we do have is a large number of examples which demand consideration as noncommutative complex spaces. These include, amongst others, noncommutative tori [1], noncommutative projective algebraic varieties [2], fuzzy flag manifolds [3], and (most importantly from the point of view of this paper) examples arising from the theory of quantum groups [4,5].

Thus far, there have been two attempts to formulate a general framework for noncommutative complex geometry. The first, due to Khalkhali, Landi, and van Suijlekom [6], was introduced to provide a context for their work on the noncommutative complex geometry of the Podleś sphere. This followed on from earlier work of Majid [5], Schwartz and Polishchuk [7], and Connes [8,9]. Khalkhali and Moatadelro [10,11] would go on to apply this framework to D'Andrea and Dąbrowski's work [12] on the higher order quantum projective spaces.

Subsequently, Beggs and Smith introduced a second more comprehensive approach to noncommutative complex geometry in [13]. Their motive was to provide a framework for quantizing the intimate relationship between complex differential geometry and complex projective geometry. They foresee that the rich interaction between algebraic and analytic techniques occurring in the classical setting will carry over to the noncommutative world.

The more modest aim of this paper is to begin the development of a theory of noncommutative complex geometry for quantum group homogeneous spaces. This will be done very much in the style of Majid's noncommutative Riemannian geometry [5,14], with the only significant difference being that here we will not need to assume that our quantum homogeneous spaces are Hopf–Galois extensions, while we will assume a faithful flatness property. This assumption allows us to use Takeuchi's categorical equivalence to establish a simple set of necessary and sufficient conditions for covariant complex

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structures to exist. In subsequent work, it is intended to build upon these results and formulate noncommutative generalizations of Hodge theory and Kähler geometry for quantum homogeneous spaces [15]. Indeed, the first steps in this direction have already been taken [16].

For this undertaking to be worthwhile, however, it will need to be applicable to a good many interesting examples. Recall that classically one of the most important classes of homogeneous complex manifolds is the family of generalized flag manifolds. As has been known for a long time, these spaces admit a direct q-deformation in terms of the Drinfeld–Jimbo quantum groups [17–19]. Somewhat more recently, it was shown by Heckenberger and Kolb [4] that the Dolbeault double complex of the irreducible flag manifolds survives this q-deformation intact. This result gives us one of the most important families of noncommutative complex structures that we have, and as such, provides an invaluable testing ground for any newly proposed theory of noncommutative complex geometry.

In this paper we show that, for the special case of quantum projective space, the work of Heckenberger and Kolb can be understood in terms of our general framework for noncommutative complex geometry. This allows for a significant simplification of the required calculations, and helps identify some of the underlying general processes at work. It is foreseen that this work will prove easily extendable to all the irreducible quantum flag manifolds. Moreover, it is hoped to extend it even further to include all the quantum flag manifolds, and in so doing, produce new examples of noncommutative complex structures.

The paper is organized as follows: In Section 2 some well-known material about quantum homogeneous spaces, Takeuchi's categorical equivalence, covariant differential calculi, almost complex structures, and complex structures is recalled.

In Section 3 the quantum special unitary group, and the quantum projective spaces, as well as the Heckenberger–Kolb calculus for these spaces, is discussed.

In Section 4 one of the basic results of the paper Proposition 4.1 is established: It shows that for a special subcategory of Mod^H_M, the monoidal structure induced on it by the canonical monoidal structure of ${}^G_M\mathrm{Mod}_M$ (through Takeuchi's equivalence) is equivalent to the vector space tensor product.

In Section 5, Theorem 5.7 shows how to find an explicit description of the maximal prolongation of a covariant first-order differential calculus in terms of a certain ideal $I^{(1)} \subseteq M^+$.

In Section 6 the notion of factorisability for almost complex structures is introduced, and a simple set of necessary and sufficient conditions for factorisable almost complex structures to exist is established.

Finally, in Section 7, Proposition 7.1 gives a simple method for verifying that an almost complex structure is a complex structure.

Throughout, the family of quantum projective spaces, endowed with the Heckenberger–Kolb calculus, is taken as the motivating set of examples. In each section, the newly constructed general theory is applied to these examples in detail, building up to an explicit presentation of their q-deformed Dolbeault double complexes.

2. Preliminaries and first results

In this section we recall Takeuchi's categorical equivalence for quantum homogeneous spaces, some of its applications to the theory of covariant differential calculi, and finally the definition of a complex structure.

2.1. Quantum homogeneous spaces

Let G be a Hopf algebra with comultiplication Δ, counit ε, antipode S, unit 1, and multiplication m. Throughout, we use Sweedler notation, as well as denoting $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for V a subspace of G. For a right G-comodule V with coaction Δ_R, we say that an element $v \in V$ is *coinvariant* if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by V^G, and call it the *coinvariant subspace* of the coaction. For H a Hopf algebra, a *homogeneous* right H-coaction on G is a coaction of the form $(\mathrm{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a Hopf algebra map. The coinvariant subspace of such a coaction is a subalgebra [20, Proposition 1].

Definition 2.1. We call the coinvariant subalgebra $M := G^H$ of a homogeneous coaction a *quantum homogeneous space* if G is faithfully flat as a right module over M, which is to say if the functor $G \otimes_M - : {}_M\mathrm{Mod} \to {}_{\mathbb{C}}\mathrm{Mod}$, from the category of left M-modules to the category of complex vector spaces, maps a sequence to an exact sequence if and only if the original sequence is exact.

In this paper we will *always* use the symbols G, H, π and M in this sense. We also note that G is itself a trivial example of a quantum homogeneous space, where $\pi = \varepsilon$. Moreover, the coproduct of G restricts to a right G-coaction on M, and

$$\pi(m) = \varepsilon(m)1_H, \quad \text{for all } m \in M. \tag{1}$$

If G and H are Hopf $*$-algebras, and π is a Hopf $*$-algebra map, then M is a $*$-subalgebra of G.