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Modular invariance and anomaly cancellation formulas in odd dimension

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ABSTRACT

By studying modular invariance properties of some characteristic forms, we get some new anomaly cancellation formulas on $(4r-1)$ dimensional manifolds. As an application, we derive some results on divisibilities of the index of Toeplitz operators on $(4r-1)$ dimensional spin manifolds and some congruent formulas on characteristic number for $(4r-1)$ dimensional spin^c manifolds.

1. Introduction

In 1983, the physicists Alvarez-Gaumé and Witten [1] discovered the "miraculous cancellation" formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch $\widehat{L}$-form and $\widehat{A}$-form of a 12-dimensional smooth Riemannian manifold. Kefeng Liu [2] established higher dimensional "miraculous cancellation" formulas for $(8k+4)$-dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas could be used to deduce some divisibility results. In [3,4], for each $(8k+4)$-dimensional smooth Riemannian manifold, a more general cancellation formula that involves a complex line bundle was established. This formula was applied to spin^c manifolds, then an analytic Ochanine congruence formula was derived. In [5], Qingtao Chen and Fei Han obtained more twisted cancellation formulas for $8k$ and $8k+4$ dimensional manifolds and they also applied their cancellation formulas to study divisibilities on spin manifolds and congruences on spin^c manifolds.

Another important application of modular invariance properties of characteristic forms is to prove the rigidity theorem on elliptic genera. For example, see [2,6–10]. For odd dimensional manifolds, we proved the similar rigidity theorem for elliptic genera under the condition that fixed point submanifolds are 1-dimensional in [11]. In [12], Han and Yu dropped off our condition and proved more general odd dimensional rigidity theorem for elliptic genera. In [12], in order to prove the rigidity theorem, they constructed some interesting modular forms under the condition that the 3th de-Rham cohomology of manifolds vanishes.

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In parallel, a natural question is whether we can get some interesting cancellation formulas in odd dimension by modular forms constructed in [12]. In this paper, we will give the confirmative answer of this question. That is, by studying modular invariance properties of some characteristic forms, we get some new anomaly cancellation formulas on $(4r-1)$ dimensional manifolds. As an application, we derive some results on divisibilities on $(4r-1)$ dimensional spin manifolds and congruences on $(4r-1)$ dimensional spin^c manifolds. In [13], a cancellation formula on 11-dimensional manifolds was derived. To the authors' best knowledge, our cancellation formulas appear for the first time for general odd dimensional manifolds.

This paper is organized as follows: In Section 2, we review some knowledge on characteristic forms and modular forms that we are going to use. In Section 3, we prove some odd dimensional cancellation formulas and we apply them to get some results on divisibilities on the index of Toeplitz operators on spin manifolds. In Section 4, we prove some odd cancellation formulas involving a complex line bundle. By these formulas, we get some congruent formulas on characteristic number for odd spin^c manifolds.

2. Characteristic forms and modular forms

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use.

2.1. Characteristic forms

Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by (cf. [14])

$$\widehat{A}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh(\frac{\sqrt{-1}}{4\pi}R^{TM})}\right),$$
$$\widehat{L}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh(\frac{\sqrt{-1}}{4\pi}R^{TM})}\right). \tag{2.1}$$

Let F, F' be two Hermitian vector bundles over M carrying Hermitian connection $\nabla^F, \nabla^{F'}$ respectively. Let $R^F = (\nabla^F)^2$ (resp. $R^{F'} = (\nabla^{F'})^2$) be the curvature of ∇^F (resp. $\nabla^{F'}$). If we set the formal difference $G = F - F'$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{F'}\right)\right]. \tag{2.2}$$

For any complex number t, let

$$\wedge_t(F) = \mathbf{C}|_M + tF + t^2\wedge^2(F) + \cdots, \qquad S_t(F) = \mathbf{C}|_M + tF + t^2S^2(F) + \cdots$$

denote respectively the total exterior and symmetric powers of F, which live in $K(M)[[t]]$. The following relations between these operations hold,

$$S_t(F) = \frac{1}{\wedge_{-t}(F)}, \qquad \wedge_t(F - F') = \frac{\wedge_t(F)}{\wedge_t(F')}. \tag{2.3}$$

Moreover, if $\{\omega_i\}$, $\{\omega'_j\}$ are formal Chern roots for Hermitian vector bundles F, F' respectively, then

$$\mathrm{ch}(\wedge_t(F)) = \prod_i (1 + e^{\omega_i}t). \tag{2.4}$$

Then we have the following formulas for Chern character forms,

$$\mathrm{ch}(S_t(F)) = \frac{1}{\prod_i (1 - e^{\omega_i}t)}, \qquad \mathrm{ch}(\wedge_t(F - F')) = \frac{\prod_i (1 + e^{\omega_i}t)}{\prod_j (1 + e^{\omega'_j}t)}. \tag{2.5}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonical induced Hermitian metric from that of W, as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W. If F is a vector bundle (complex or real) over M, set $\widetilde{F} = F - \dim F$ in $K(M)$ or $KO(M)$.

2.2. Some properties about the Jacobi theta functions and modular forms

We first recall the four Jacobi theta functions are defined as follows (cf. [15]):