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Extendability of parallel sections in vector bundles

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ABSTRACT

I address the following question: Given a differentiable manifold M, what are the open subsets U of M such that, for all vector bundles E over M and all linear connections ∇ on E, any ∇-parallel section in E defined on U extends to a ∇-parallel section in E defined on M?

For simply connected manifolds M (among others) I describe the entirety of all such sets U which are, in addition, the complement of a C^1 submanifold, boundary allowed, of M. This delivers a partial positive answer to a problem posed by Antonio J. Di Scala and Gianni Manno (2014). Furthermore, in case M is an open submanifold of $\mathbb{R}^n$, $n \geq 2$, I prove that the complement of U in M, not required to be a submanifold now, can have arbitrarily large n-dimensional Lebesgue measure.

1. Introduction

In their recent article Antonio J. Di Scala and Gianni Manno raise the following question [1, Problem 1]: given a vector bundle E over a simply connected manifold M, a connection ∇ on E, and a ∇-parallel section σ in E defined on an open, dense, connected subset $U \subset M$, does there exist a ∇-parallel section $\tilde{\sigma}$ defined on M such that $\tilde{\sigma}$ extends σ—that is, such that $\tilde{\sigma}|_U = \sigma$? Di Scala and Manno explain, among others, how to apply this question to the extension of Killing vector fields. Please consult their introduction for details as well as further applications.

For the note at hand, I would like to widen the scope of Di Scala's and Manno's question slightly suggesting an alternative problem: for a given manifold M (simply connected or not), describe/characterize the set of all open subsets $U \subset M$ such that, for all vector bundles E over M, all connections ∇ on E, and all ∇-parallel sections σ in E defined on U, there exists a ∇-parallel extension $\tilde{\sigma}$ as above. As a matter of fact, I will try and characterize the universe of closed subsets $F \subset M$ whose complement $U = M \backslash F$ has the aforementioned property. These closed subsets $F \subset M$ will be called *negligible* in M (see Definition 4.1).

My results come in two groups. For one thing, in Section 4, I derive *necessary conditions* for a set F to be negligible in M. Specifically, I prove that when F is negligible in M, and M is connected, then the complement $M \backslash F$ is necessarily connected too (Proposition 4.8). When M is of dimension 2 or higher, then, moreover, F needs to be nowhere dense in M (Corollary 4.11). Observe that Di Scala and Manno have already pointed these two conditions out as relevant—without, however, proving their necessity [1].

For another, in Section 5, I derive *sufficient conditions* for a set F to be negligible in M. This is probably the more interesting part (as compared to Section 4) since here I prove that parallel extensions of parallel sections do in fact exist. The most

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striking result of Section 5 is Corollary 5.11 which asserts in particular (compare Remark 5.12) that when M is a simply connected (second-countable, Hausdorff) manifold and $F \subset M$ is a closed C^1 submanifold with boundary such that $M \backslash F$ is dense and connected in M, then F is negligible in M. Hence, Corollary 5.11 yields a partial positive answer for the question of Di Scala and Manno.

As a sideline in Section 5, I show that the Lebesgue measure of a set F is quite unrelated to the negligibility of F. Precisely, I prove the existence of negligible subsets of $\mathbb{R}^n$ of arbitrarily large, and even infinite, measure (Corollary 5.6). Besides, and contrasting, I show that the fact that F has Lebesgue measure 0 inside $\mathbb{R}^n$, $n \geq 2$, does not imply that F is negligible in $\mathbb{R}^n$ for all connections of class C^0, even when $\mathbb{R}^n \backslash F$ is connected (Corollary 5.4). The latter observation tells us that the smoothness of the connection is essential in Di Scala's and Manno's question.

Sections 2 and 3 contain preliminary definitions, conventions, and remarks that I employ in the course of Sections 4 and 5.

2. Manifolds and submanifolds with boundary

By a *manifold* I mean a locally finite-dimensional (i.e., locally modeled on some $\mathbb{R}^n$, $n \in \mathbb{N}$) differentiable manifold of class C^k, $1 \leq k \leq \infty$, without boundary; I make no topological assumptions whatsoever (cf. [2, p. 23]). The extended (i.e., allowing ∞) natural number k will be fixed throughout. The sheaf of real-valued functions of class C^m on M, $0 \leq m \leq k$, will be denoted by C^m_M, or by C^m when M is clear from the context.

Let us recall, mainly for the sake of Theorem 5.9, some terminology concerning manifolds and submanifolds with boundary.[1]

Definition 2.1. Let M be a manifold, $F \subset M$ a subset, $0 \leq m \leq k$ a natural number or ∞.

Let $p \in F$. Then F is a C^m *submanifold with boundary* of M at p when there exist $d, c \in \mathbb{N}$, an open neighborhood U of p in M, an open subset V of $\mathbb{R}^d \times \mathbb{R}^c$, and an isomorphism $\phi\colon U \to V$ of class C^m such that

$$\phi(F \cap U) = (H_d \times \{(0, \dots, 0)\}) \cap V,$$

where

$$H_d := \begin{cases} \{(x_1, \dots, x_d) \in \mathbb{R}^d : 0 \leq x_d\} & \text{when } d > 0, \\ \mathbb{R}^0 & \text{when } d = 0. \end{cases} \tag{2.1.1}$$

In that case we write

$$\operatorname{codim}_p(F, M) = c.$$

We say that F is a C^m *submanifold with boundary* of M if F is a C^m submanifold with boundary of M at p for all $p \in F$. If this is the case, we set

$$\operatorname{codim}(F, M) := \inf\{\operatorname{codim}_p(F, M) : p \in F\},$$

where the infimum of the empty set is taken to be ∞.

In Section 5 we are interested in nowhere dense, closed subsets F of manifolds M. In case F is a submanifold with boundary of M, nowhere density can be characterized in terms of the codimension of F inside M.

Remark 2.2 (*Density and Codimension*)**.** Let M be a manifold, F a closed C^0 submanifold with boundary of M. Then the following are equivalent:

(1) F is nowhere dense in M.
(2) $1 \leq \operatorname{codim}(F, M)$.

Assume item 1. Let $p \in F$. Assume $\operatorname{codim}_p(F, M) = 0$. Then there exists an open neighborhood U of p in M, a number $n \in \mathbb{N}$, an open subset $V \subset \mathbb{R}^n$, and a homeomorphism $\phi\colon U \to V$ such that $\phi(F \cap U) = H_n \cap V$. If $n = 0$, then $U = \{p\} \subset F$. Thus p is an interior point of F in M, contradicting item 1. If $n \geq 1$, then there exists a number $\epsilon > 0$ such that

$$y := \phi(p) + (0, \dots, 0, \epsilon) \in \{x \in \mathbb{R}^n : 0 < x_n\} \cap V.$$

Thus the preimage of the point y is an interior point of F in M—contradiction. So, $1 \leq \operatorname{codim}_p(F, M)$. As $p \in F$ was arbitrary, we have item 2.

Assume item 2, and let $p \in F$. Then there exists an open neighborhood U of p in M, numbers $d, c \in \mathbb{N}$, an open subset V of $\mathbb{R}^d \times \mathbb{R}^c$, and a homeomorphism $\phi\colon U \to V$ such that $\phi(F \cap U) \subset (\mathbb{R}^d \times \{(0, \dots, 0)\}) \cap V$. If p were an interior point of F in M, the point $\phi(p)$ would be an interior point of $\mathbb{R}^d \times \{(0, \dots, 0)\}$ in $\mathbb{R}^d \times \mathbb{R}^c$. Yet as $1 \leq \operatorname{codim}(F, M) \leq \operatorname{codim}_p(F, M) = c$, this is absurd.

[1] The manifolds with boundary that we consider will always arise as submanifolds.