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Hypersurfaces in space forms satisfying some curvature conditions

Ryszard Deszcz [a,*], Małgorzata Głogowska [a], Marian Hotloś [b], Georges Zafindratafa [c]

[a] Department of Mathematics, Wrocław University of Environmental and Life Sciences, Grunwaldzka 53, 50-357 Wrocław, Poland

[b] Department of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

[c] Laboratoire de Mathématiques et Applications de Valenciennes, Université de Valenciennes et du Hainaut-Cambrésis, Le Mont Houy, Lamath-ISTV2, 59313 Valenciennes cedex 9, France

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ABSTRACT

In Abdalla and Dillen (2002) an example of a non-semisymmetric Ricci-symmetric quasi-Einstein austere hypersurface M isometrically immersed in an Euclidean space was constructed. In this paper we state that, at every point of the hypersurface M, the following generalized Einstein metric curvature condition is satisfied: $(*)$ the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(S, C)$ are linearly dependent. Precisely, $(n-2)(R \cdot C - C \cdot R) = Q(S, C)$ on M. We also prove that non-conformally flat and non-Einstein hypersurfaces with vanishing scalar curvature having at every point two distinct principal curvatures, as well as some hypersurfaces having at every point three distinct principal curvatures, satisfy $(*)$. We present examples of hypersurfaces satisfying $(*)$.

1. Introduction

In [1] a survey on some family of generalized Einstein metric conditions was given. Those curvature conditions are strongly related to pseudosymmetry. In particular, [1, Section 6] contains results of non-Einstein and non-conformally flat semi-Riemannian manifolds (M, g), of dimension $n \geq 4$, satisfying conditions of the form: the tensor $R \cdot C - C \cdot R$ is proportional to the Tachibana tensor: $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ or $Q(S, C)$. More precisely, we consider those conditions on the set $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ of all points at which the Ricci tensor S is not proportional to the metric tensor g (the set $\mathcal{U}_S$) and

* Corresponding author.
E-mail addresses: Ryszard.Deszcz@up.wroc.pl (R. Deszcz), Malgorzata.Glogowska@up.wroc.pl (M. Głogowska), Marian.Hotlos@pwr.edu.pl (M. Hotloś), Georges.Zafindratafa@univ-valenciennes.fr (G. Zafindratafa).

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the Weyl conformal curvature tensor C is non-zero (the set $\mathcal{U}_C$). Among other results in that section, it was shown that some hypersurfaces M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfy $(*)$, i.e. the condition

$$R \cdot C - C \cdot R = L\, Q(S, C), \tag{1.1}$$

where L is some function on $\mathcal{U}_S \cap \mathcal{U}_C$. We recall that an example of a hypersurface having mentioned properties was constructed in [2, Section 5]. We also mention that semi-Riemannian manifolds satisfying (1.1) were recently investigated in [3]. For precise definitions of the symbols used here, we refer to Sections 2–4. Those sections also contain some preliminary results.

Let M be a hypersurface isometrically immersed in $N_s^{n+1}(c)$, $n \geq 4$. We denote by $\mathcal{U}_H \subset M$ the set of all points at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H. We can verify that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C \subset M$ (see, e.g., [4, p. 137]). In Section 5 we consider hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.1) on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$. We note that on this set we have

$$H^2 = \alpha H + \beta g. \tag{1.2}$$

According to [4, Proposition 3.3], the Riemann–Christoffel curvature tensor R of M is expressed on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$ by a linear combination of the Kulkarni–Nomizu products $S \wedge S$, $g \wedge S$ and $G = \frac{1}{2}\, g \wedge g$ formed by the Ricci tensor S and the metric tensor g of M. Precisely, we have

$$R = \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \eta\, G, \tag{1.3}$$

where ϕ, μ and η are some functions on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H$ (see (5.2)). We can also express the tensors $C \cdot C$, $Q(g, C)$ and $Q(S, C)$ by some linear combinations of the Tachibana tensors formed by the tensors g and H. In Theorem 5.1 we state that if the scalar curvature κ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then

$$R \cdot C - C \cdot R = -Q(S, C) \tag{1.4}$$

on this set. From that theorem it follows immediately (see, Corollary 5.3) that if M is a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point exactly two distinct principal curvatures, and if its scalar curvature κ vanishes on $(\mathcal{U}_S \cap \mathcal{U}_C) \setminus \mathcal{U}_H \subset M$ then (1.4) holds on this set. In Examples 5.4, 5.5 and 5.7 we present examples of non-conformally flat and non-Einstein hypersurfaces, with $\kappa = 0$, having at every point exactly two distinct principal curvatures.

As it was stated in [5, Corollary 4.1], for a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, if at every point of $\mathcal{U}_H \subset M$ one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ is a linear combination of the tensor $R \cdot R$ and a finite sum of the Tachibana tensors of the form $Q(A, B)$, where A is a symmetric $(0, 2)$-tensor and B a generalized curvature tensor, then on $\mathcal{U}_H$

$$H^3 = \mathrm{tr}(H)\, H^2 + \psi\, H + \rho\, g, \tag{1.5}$$

where ψ and ρ are some functions on this set. Thus in particular, if (1.1) is satisfied on $\mathcal{U}_H$ then (1.5) holds on this set. Hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.5), or in particular (1.5) with $\rho = 0$, i.e.

$$H^3 = \mathrm{tr}(H)\, H^2 + \psi\, H, \tag{1.6}$$

were investigated in several papers: [6–8,5,9,2,10–21]. Section 6 contains some results on hypersurfaces satisfying (1.5). In Section 7 we consider hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1.1) on $\mathcal{U}_H \subset M$. The main result of this section (see, Theorem 7.3) states that if M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ the equalities: (1.6) and

$$\mathrm{rank}\,(S - \alpha\, g) = 1, \tag{1.7}$$

for some function α on $\mathcal{U}_H$, then on this set

$$\mathrm{rank}\left(S - \left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n(n+1)}\right) g\right) = 1, \tag{1.8}$$

$$(n-2)\,(R \cdot C - C \cdot R) = Q(S, C) - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, C). \tag{1.9}$$

In particular, if the ambient space is a semi-Euclidean space $\mathbb{R}_s^{n+1}$, $n \geq 4$, then on $\mathcal{U}_H$

$$\mathrm{rank}\left(S - \frac{\kappa}{n-1}\, g\right) = 1, \tag{1.10}$$

$$(n-2)\,(R \cdot C - C \cdot R) = Q(S, C). \tag{1.11}$$

Let M be a hypersurface in an Euclidean space $\mathbb{R}^{n+1}$, $n = 2p + 1$, $p \geq 2$, having at every point three principal curvatures $\lambda_1 = \lambda \neq 0$, $\lambda_2 = -\lambda$ and $\lambda_3 = 0$, provided that the multiplicity of λ_1, as well as of λ_2 is p. Clearly, M is an austere