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Existence and classification of three-dimensional Lorentzian manifolds with prescribed distinct Ricci eigenvalues

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ABSTRACT

We prove that there exists at least one three-dimensional Lorentzian manifold with any prescribed distinct Ricci eigenvalues, which are given as arbitrary real analytic functions. Moreover, we prove that such Lorentzian manifolds depend locally on three arbitrary functions of two variables.

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Introduction

There is a well-known fundamental result by DeTurck [1] about existence of metrics with prescribed Ricci curvature. Here we shall handle a related but different problem concerning prescribed Ricci curvature operator.

The problem of how many Riemannian metrics exist on the open domains of $\mathbb{R}^3$ with prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ was completely solved in [2] and [3]. The main existence theorem says that the local isometry classes of these metrics are always parametrized by two-arbitrary functions of one variable. Some non-trivial explicit examples were presented in [2], as well. A more elegant but less rigorous proof of the main existence theorem was given in [4].

The case of distinct constant Ricci eigenvalues is more interesting. Here, the first examples were presented by K. Yamato in [5], namely a complete (but not locally homogeneous) Riemannian metric defined on $\mathbb{R}^3$ for each prescribed triplet of constant distinct Ricci eigenvalues satisfying certain algebraic inequalities. Thus, these triplets form an open set in $\mathbb{R}^3[\varrho_1.\varrho_2, \varrho_3]$. This open set was essentially extended by new examples in [6]. Finally, in [7], non-trivial explicit examples were constructed for *every choice* of Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. This result was extended successfully by G. Calvaruso to the Lorentzian case (see [8]). Some related references are [9–12].

The problem of "*how many*" local isometry classes (or, more exactly, how many isometry classes of germs) of Riemannian metrics exist for prescribed constant Ricci eigenvalues was solved first by A. Spiro and F. Tricerri in [13], using the theory of formally integrable analytic differential systems. They proved that this "local moduli space" depends on an infinite number of

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parameters. This solution was not satisfactory enough for the first author and Z. Vlášek, and we succeeded to show in [14] that this local moduli space is parametrized, in fact, by (the germs of) three-arbitrary functions of two variables. Moreover, the method of solution was completely "classical", based on the Cauchy–Kowalevski Theorem. Yet, for many mathematicians, this solution might be not completely satisfactory for a different reason: The partial differential equations expressing the geometric conditions are rather cumbersome and one of the main steps of the proof is not transparent enough, because it depends heavily on a hard computer work (using Maple V) for the huge amount of routine symbolic manipulations with the corresponding system of partial differential equations.

Therefore, in the next paper [15], the first author and Vlášek tried to prove the same result by a different method under more general assumptions for the Ricci eigenvalues. Namely, the prescribed Ricci eigenvalues are here not constants but arbitrary real analytic functions. Unfortunately, after the publication of the paper the authors found a basic mistake, namely the relations $\mathrm{Ric}^i_j = \mathrm{Ric}^j_i$ for $i, j = 1, 2, 3$ which are not true. Thus we have to annul this part of paper [15], not the paper itself because it contains other results of interest, which are correct.

The present paper was originally intended to correct the proof from [15]. But it appeared that the modification to the Lorentzian case is easier to perform, and this is the real aim of the present paper. As concerns the Riemannian case, it seems to be more difficult and the work it is still in preparation.

We use *Mathematica* 6 for calculations.

1. A special case of the Cauchy–Kowalevski theorem

We shall recall first the classical version of the Cauchy–Kowalevski Theorem (see [16]). The more modern versions can be found in [17]. For our purposes, we shall need just a very special case of this well-known Theorem.

Theorem 1. *Consider the system of partial differential equations of second order for the unknown functions* $U^i = U^i(x^1, x^2, x^3)$, $i = 1, 2, 3$*, as follows:*

$$\begin{aligned} \frac{\partial^2 U^1}{(\partial x^3)^2} &= H^1\Big(x^i, U^i, \frac{\partial U^i}{\partial x^j}, \frac{\partial^2 U^i}{\partial x^k \partial x^l}\Big), \\ \frac{\partial^2 U^2}{(\partial x^3)^2} &= H^2\Big(x^i, U^i, \frac{\partial U^i}{\partial x^j}, \frac{\partial^2 U^i}{\partial x^k \partial x^l}\Big), \\ \frac{\partial^2 U^3}{(\partial x^3)^2} &= H^3\Big(x^i, U^i, \frac{\partial U^i}{\partial x^j}, \frac{\partial^2 U^i}{\partial x^k \partial x^l}\Big), \end{aligned} \tag{1}$$

where H^i*'s are real analytic functions of their variables in a neighborhood* $\mathcal{B} \subset \mathbb{R}^3[x^1, x^2, x^3]$ *of a point* $o = (x^1_0, x^2_0, x^3_0)$, $j = 1, 2, 3$ *and* $1 \le k \le l \le 3$, $(k, l) \ne (3, 3)$*. Let* $M^i_0 = M^i_0(x^1, x^2)$ *and* $M^i_1 = M^i_1(x^1, x^2)$, $i = 1, 2, 3$*, be real analytic functions in a sufficiently small neighborhood* $\mathcal{B}'$ *of the point* $o' = (x^1_0, x^2_0)$ *in the* (x^1, x^2)*-plane. Then there is a unique triplet* $\big(U^1(x^1, x^2, x^3), U^2(x^1, x^2, x^3), U^3(x^1, x^2, x^3)\big)$ *which is the solution of the system* (1) *and satisfies* $U^i(x^1, x^2, x^3_0) = M^i_0(x^1, x^2)$ *and* $(\partial U^i/\partial x^3)(x^1, x^2, x^3_0) = M^i_1(x^1, x^2)$, $i = 1, 2, 3$*, in* $\mathcal{B}'$.

Further, denote

$$\begin{aligned} a^i &= U^i(x^1_0, x^2_0, x^3_0) = M^i_0(x^1_0, x^2_0), \\ a^i_j &= \frac{\partial U^i}{\partial x^j}(x^1_0, x^2_0, x^3_0), \\ a^i_{kl} &= \frac{\partial^2 U^i}{\partial x^k \partial x^l}(x^1_0, x^2_0, x^3_0), \end{aligned}$$

for $i, j = 1, 2, 3$ and $1 \le k \le l \le 3$, $(k, l) \ne (3, 3)$. It is clear that a^i, a^i_j and a^i_{kl} can be chosen as arbitrary constants (in fact, as certain coefficients of the Taylor polynomials of a given solution (U^1, U^2, U^3), which we call Cauchy initial constants).

Convention: Instead of the upper index i, we shall now write the letters a, b and c for $i = 1, 2, 3$ respectively. Thus the Cauchy initial constants will be denoted as $a, b, c, a_j, b_j, c_j, a_{kl}, b_{kl}, c_{kl}$, where $(k, l) \ne (3, 3)$.

2. The main theorem

We shall start with an auxiliary result which simplifies essentially the computations, and which was used, in the Riemannian case, already in Kowalski–Vlášek [15].

Theorem 2 (*Diagonalization Theorem* [16])**.** *Let* (M, g) *be a real analytic three-dimensional Lorentzian manifold. Then, in a neighborhood of each point* $p \in M$*, there is a system* (x, y, z) *of local coordinates in which* g *adopts a diagonal form. All coordinate transformations for which the diagonality of* g *is preserved depend (locally) on three arbitrary real analytic functions of two variables.*

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