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Cauchy–Schwarz-type inequalities on Kähler manifolds-II

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Dedicated to Professor Martin Guest on the occasion of his 60th birthday

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ABSTRACT

We establish in this note some Cauchy–Schwarz-type inequalities on compact Kähler manifolds, which generalize the classical Khovanskii–Teissier inequalities to higher-dimensional cases. Our proof is to make full use of the mixed Hodge–Riemann bilinear relations due to Dinh and Nguyên. A proportionality problem related to our main result is also proposed.

1. Introduction and main results

Suppose X is an n-dimensional algebraic manifold and $D_1, D_2, \dots, D_n$ are n (not necessarily distinct) ample divisors on X. Then we have the following opposite Cauchy–Schwarz-type inequality,

$$([D_1D_2D_3\cdots D_n])^2 \geq [D_1D_1D_3\cdots D_n]\cdot[D_2D_2D_3\cdots D_n], \tag{1.1}$$

where $[\cdot]$ denotes the intersection number of the divisors inside it and the equality holds if and only if the two divisors D_1 and D_2 are numerically proportional.

(1.1) was discovered independently by Khovanskii and Teissier around in 1979 [1,2] and now is called Khovanskii–Teissier inequality. This equality is indeed a generalization of the classical Aleksandrov–Fenchel inequalities and thus present a nice relationship between the theory of mixed volumes and algebraic geometry [3, p. 114]. The proof of (1.1) is to apply the usual Hodge–Riemann bilinear relations [4, p. 122–123] to the Kähler classes determined by these divisors and an induction argument. The approach also suggests that the usual Hodge–Riemann bilinear relations may be extended to the mixed case. After some partial results towards this direction [5,6], this aim was achieved in its full generality by Dinh and Nguyên in [7].

We would like to point out a fact, which was not mentioned explicitly in [7], that (1.1) can now be extended by the mixed Hodge–Riemann bilinear relations as follows. Suppose $\omega, \omega_1, \omega_2, \dots, \omega_{n-2}$ are $n-1$ Kähler classes and $\alpha \in H^{1,1}(M, \mathbb{R})$ an *arbitrary* real-valued $(1, 1)$-form on an n-dimensional compact connected Kähler manifold M. Then we have

$$\left(\int_M \alpha\wedge\omega\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\right)^2 \geq \left(\int_M \alpha^2\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\right)\cdot\left(\int_M \omega^2\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\right), \tag{1.2}$$

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where the equality holds if and only if $\alpha \in \mathbb{R}\omega$. Indeed, [7, Theorem A] tells us that the index of the following bilinear form

$$Q(u,v) := \int_M u \wedge v \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}, \quad u, v \in H^{1,1}(M, \mathbb{R}), \tag{1.3}$$

is of the form $(+, -, \ldots, -)$, i.e., the positive and negative indices are 1 and $h^{1,1} - 1$ respectively, where $h^{1,1}$ is the corresponding Hodge number of M (the dimension of $H^{1,1}(M, \mathbb{R})$). Define a real-valued function $f(t) := Q(\omega + t\alpha, \omega + t\alpha) (t \in \mathbb{R})$. Then $f(0) > 0$ as $\omega, \omega_1, \ldots, \omega_{n-2}$ are all Kähler classes and thus their product is strictly positive. $\omega + t\alpha$ spans a 2-dimensional subspace in $H^{1,1}(M, \mathbb{R})$ if α and ω are linearly independent and thus $f(t_0) < 0$ for some $t_0 \in \mathbb{R}$ in view of the index of $Q(\cdot, \cdot)$. Then the discriminant of $f(t)$ gives (1.2) with strict sign " $>$".

When these Kähler classes are all equal: $\omega = \omega_1 = \cdots = \omega_{n-2}$, (1.2) degenerates to the following special case:

$$\left(\int_M \alpha \wedge \omega^{n-1}\right)^2 \geq \left(\int_M \alpha^2 \wedge \omega^{n-2}\right) \cdot \left(\int_M \omega^n\right), \quad \forall\, \alpha \in H^{1,1}(M, \mathbb{R}), \tag{1.4}$$

which is quite well-known and, to the author's best knowledge, should be due to Apte in [8]. Inspired by (1.4), the author asked in [9] whether or not there exists a similar inequality to (1.4) for those $\alpha \in H^{p,p}(M, \mathbb{R})$ $(1 \leq p \leq [\frac{n}{2}])$ and obtained a related result [9, Theorem 1.3], whose proof is also based on the usual Hodge–Riemann bilinear relations. As an application we presented some Chern number inequalities when the Hodge numbers of the manifolds satisfy some constraints [9, Corollary 1.5]. Now keeping the mixed Hodge–Riemann bilinear relations established in [7] in mind, we may also ask if the main idea of the proof in [9] can be carried over to the mixed case to extend the α in (1.2) to $H^{p,p}(M, \mathbb{R})$ for $1 \leq p \leq [\frac{n}{2}]$. The answer is affirmative and this is the main goal of our current article. So this article can be viewed as a sequel to [9], which explains its title either.

Our main result (Theorem 1.3) will be stated in the rest of this section. In Section 2 we briefly review the mixed Hodge–Riemann bilinear relations and then present the proof of Theorem 1.3. In Section 3 we discuss a proportionality problem related to (1.1) posed by Teissier and propose a similar problem related to our main result.

In order to state our result as general as possible, we would like to investigate the elements in $H^{p,p}(M, \mathbb{C})$, i.e., complex-valued (p, p)-forms on M. The following definition is inspired by (1.2) and is a mixed analogue to [9, Definition 1.1].

Definition 1.1. Suppose M is an n-dimensional compact connected Kähler manifold. For $1 \leq p \leq [\frac{n}{2}]$, $\alpha \in H^{p,p}(M, \mathbb{C})$ and $n - 2p + 1$ Kähler classes $\omega, \omega_1, \ldots, \omega_{n-2p}$, we put $\Omega_p := \omega_1 \wedge \cdots \wedge \omega_{n-2p}$ and define

$$g(\alpha, \omega; \Omega_p) := \left(\int_M \alpha \wedge \bar{\alpha} \wedge \Omega_p\right) \cdot \left(\int_M \omega^{2p} \wedge \Omega_p\right) - \left(\int_M \alpha \wedge \omega^p \wedge \Omega_p\right) \cdot \left(\int_M \bar{\alpha} \wedge \omega^p \wedge \Omega_p\right).$$

α is said to satisfy Cauchy–Schwarz (resp. opposite Cauchy–Schwarz) inequality with respect to the Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$ if $g(\alpha, \omega; \Omega_p) \geq 0$ (resp. $g(\alpha, \omega; \Omega_p) \leq 0$).

Remark 1.2. Note that $\alpha \in H^{p,p}(M, \mathbb{R})$ if and only if $\alpha = \bar{\alpha}$. Also note that $g(\alpha, \omega; \Omega_p)$ in the above definition is a real number and so we can discuss its non-negativity or non-positivity.

The main result of this note, which extends [9, Theorem 1.3] to the mixed case, is the following:

Theorem 1.3. *Suppose M is an n-dimensional compact connected Kähler manifold.*

(1) *Given $1 \leq p \leq [\frac{n}{2}]$, all elements in $H^{p,p}(M, \mathbb{C})$ satisfy Cauchy–Schwarz inequality with respect to any Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$ (in the sense of Definition 1.1) if and only if the Hodge numbers of M satisfy*

$$h^{2i,2i} = h^{2i+1,2i+1}, \quad 0 \leq i \leq \left[\frac{p+1}{2}\right] - 1. \tag{1.5}$$

(2) *All elements in $H^{1,1}(M, \mathbb{C})$ satisfy opposite Cauchy–Schwarz inequality with respect to any Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$.*

(3) *Given $2 \leq p \leq [\frac{n}{2}]$, all elements in $H^{p,p}(M, \mathbb{C})$ satisfy opposite Cauchy–Schwarz inequality with respect to any Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$ if and only if the Hodge numbers of M satisfy*

$$h^{2i-1,2i-1} = h^{2i,2i}, \quad 1 \leq i \leq \left[\frac{p}{2}\right]. \tag{1.6}$$

Moreover, in all the cases mentioned above, the equalities hold if and only if these α are proportional to ω^p.

The first part of the following corollary extends the Khovanskii–Teissier inequalities (1.1) and (1.2).