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A dimension associated with a cutting of the square of a Gibbs measure

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ABSTRACT

Given a compact Riemann manifold M. let $F: M \to M$ be a diffeomorphism and let μ be an F invariant ergodic measure. In [6] (Ledrappier and Young, 1985), Ledrappier and Young have proved that μ is exact dimensional. We propose to give a direct proof of this result when μ is a Gibbs measure, defined on a symbolic space product $\sum_{r_1} \times \sum_{r_2}$ with $2 \leqslant r_1 < r_2$ integers, and invariant by the shift.

1. Introduction

The present paper addresses the problem of giving formulas for the Hausdorff dimension of a self similar set when the self-mapping is not conformal, but has different exponents. Notice that in [6], Ledrappier and Young gave a general result for any F ergodic measure μ, where F is C^2 diffeomorphism of a compact Riemannian manifold. Besides, Shu [9], has recently treated the case for C^2 non degenerate endomorphism on a compact Riemannian manifold.

Here the model is a symbolic shift on a product space.We denote by $\sum_r = \{0, 1, \dots r-1\}^{\mathbb{N}}$, for $r \geqslant 2$ integer, and we consider the symbolic space product $\sum_{r_1} \times \sum_{r_2}$, endowed with the metric product. For dimensional purposes, we can equivalently consider the mapping $F: [0,1[\times[0,1[\to [0,1[\times[0,1[$ defined by $F(x,y) = (r_1 x (mod 1), r_2 y (mod 1))$ with $2 \leqslant r_1 < r_2$ integers.

It follows from the theorem C', in [6], that the measure μ is exact dimensional, i.e; there is δ such that, for μ-a.e (x,y)

$$\lim_{\varepsilon \to 0} \frac{\ln \mu((x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon))}{\ln \varepsilon} = \delta \tag{1}$$

where

$$\delta = \frac{h(\mu)}{\ln r_2} + \left(1 - \frac{\ln r_1}{\ln r_2}\right)\gamma \tag{2}$$

while noting that $h(\mu)$ is the entropy of the measure μ and γ is the dimension of the projection of μ on $\sum_{r_1}$.

Really, the reference [6] is much more general and therefore not so easy to grasp. However, we suggest giving a direct proof when the setting is much simpler, in particular for Gibbs measure.

More precisely, we determine the Hausdorff dimension of the measure μ associated with a potential ϕ on the space product $\sum_{r_1} \times \sum_{r_2}$ endowed with the ultrametric distance, and thus μ is constructed on the rectangles that flatten as their diameter tends to zero. However, the rectangles do not allow the calculation of the Hausdorff dimension, hence the difficulty of the problem. To solve such a problem, we will use two main tools which are on the one hand, an important result known as the Shannon–McMillan–Brieman theorem [7], which asserts that for an ergodic transformation-preserving measure, the limit as n goes to ∞ of the logarithm of the mass of an n-cylinder by n, exists and equals its entropy and on the other hand, the essential result of Yanick Heurteaux [5], which states that for a quasi-Bernoulli measure μ, the L^q-spectrum τ of μ is differentiable at 1. Besides, $\tau'(1)$ exists and equals the limit, as n goes to ∞ of the logarithm of the mass of an n-cylinder by

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n. It is worth to emphasize that the present work joins within the frame of several studies done in [1,3,8].

We organize the materials in this paper as follows: in the next section, we give some definitions and notations which will be useful, and we state our main results,and in the third section we provide the proof of the Theorem 1.

2. Notations and main results

For $r \geqslant 2$ integer, we denote by $\sum_r$ denote the symbolic space $\{0,1,\ldots,r-1\}^{\mathbb{N}}$. $\sum_r$ is also called the set of the one-sided sequences $z=(z_i)_{i\geqslant 1}$. $\sum_r^n$ stands for the set of finite words of length n.

The sets $\sum_r$ and $\sum_r^n$ are respectively endowed with the concatenation operation which is defined as follows: if $\omega \in \sum_r^n$ and $\omega' \in \sum_r$ (resp. $\omega \in \sum_r^n$ and $\omega' \in \sum_r^m$) then $\omega\omega'$ denotes the word obtained by juxtaposition of ω and ω'.

If $z = z_1\, z_2 \ldots z_p \ldots \in \sum_r$, then $z|n$ stands for the prefix $z_1\, z_2 \ldots z_n$ of z. We define the n-cylinder of $\sum_r$ as follows: for $\omega = \omega_1 \ldots \omega_n \in \sum_r^n$, let

$$I(\omega) = \left\{z \in \sum\nolimits_r ;\, z|n = \omega\right\} \tag{3}$$

and for $z \in \sum_r$,

$$I_n(z) = I(z|n) \tag{4}$$

The set $\sum_r$ is endowed with the metric d defined as follows: for $r \geqslant 2$ integer

$$d : (z,z') \in \sum\nolimits_r \times \sum\nolimits_r \mapsto r^{-n} \tag{5}$$

where $n = \inf\left\{p \geqslant 1,\, z_p \neq z'_p\right\}$and $d(z,z') = 0$ if $z = z'$.

Given a borelian measure μ on $\sum_r$. The entropy of μ is defined in [10] by:

$$h(\mu) = \lim_{n\to+\infty} \frac{-1}{n} \sum_{\omega\in\sum_r^n} \mu(I(\omega)) \ln \mu(I(\omega)) \tag{6}$$

whenever this limit exists.

We denote by σ denotes the shift transformation. It is defined on $\sum_r$ as follows:

$$\sigma : (z_n)_{n\geq 1} \in \sum\nolimits_r \mapsto (z_{n+1})_{n\geq 1}$$

Let $\phi : \sum_r \longrightarrow \mathbb{R}$ be a Hölder function i.e; there exist $\eta \in (0,1]$ and $L > 0$ such that:

$$|\phi(z) - \phi(z')| \leq L d(z,z')^{\eta}; \quad \forall\ (z,z') \in \sum\nolimits_r \times \sum\nolimits_r$$

Bowen showed in [6] that if ϕ is a Hölder function then the following limit

$$P(\phi) = \lim_{n\to\infty} \frac{\ln\left(\sum_{z_1\ldots z_n} \exp S_n(\phi)(z')\right)}{n}; \quad z' \in I(z_1\ldots z_n)$$

called the pressure of ϕ, exists. Where $S_n(\phi)(z) = \sum_{i=0}^{n-1}\phi(\sigma^i z)$ and $z \in \sum_r$.

Now, we can pronounce the following result, due to Bowen [4], which concerns the existence of Gibbs measure.

Theorem 4. *Suppose that $\phi : \sum_r \longrightarrow \mathbb{R}$ is a Hölder function. There exists a unique σ-invariant measure μ_ϕ such that:*

$$c \leqslant \frac{\mu_\phi(I_n(z))}{\exp\left(-nP + \sum_{i=0}^{n-1}\phi(\sigma^i z)\right)} \leqslant c^{-1} \tag{7}$$

$\forall z \in \sum_r$, $\forall n \geqslant 1$, where $c > 0$ and P is the pressure of ϕ. The measure μ_ϕ is called the Gibbs measure corresponding to ϕ.

Let $2 \leqslant r_1 < r_2$ be two integers. for $i \in \{1,2\}$, recall that $\sum_{r_i}$ denotes the set $\{0,1,\ldots\ldots r_{i-1}\}^{\mathbb{N}}$. We mention that $\sum_r$ is identified to $\sum_{r_1} \times \sum_{r_2}$. For $X = (x,y) \in \sum_r$, we have $I_n(X) = I_n(x) \times I_n(y)$. $I_n(X)$ is called the rectangle of $\sum_r$ of nth generation containing X. If ϕ is a potential defined on $\sum_r$ and μ_ϕ is a measure as the one defined in (7), we recall that

$$dim\left(\mu_\phi\right) = \frac{h\left(\mu_\phi\right)}{\ln\,(r_1 r_2)}$$

In what follows, we are going to be interested in the measure μ_ϕ constructed on rectangles and we suggest determining its Hausdorff dimension when the space $\sum_r$ is endowed with the metric product and not the ultradistance. For the need, we introduce the following quantity.

For every integer n, we denotes by $q(n)$ the integer such that:

$$n\frac{\ln r_2}{\ln r_1} \leqslant q(n) < n\frac{\ln r_2}{\ln r_1} + 1$$

Then, for $X = (x,y) \in \sum_r$, let $C_n(X) = I_{q(n)}(x) \times I_n(y)$. $C_n(X)$ stands for the "almost-square" of $\sum_r$ of nth generation containing X.

Theorem 1.

$$\lim_{n\to\infty}\frac{1}{n}\ln \mu_\phi(C_n(X)) = -h\left(\mu_\phi\right) + \left(\frac{\ln r_2}{\ln r_1} - 1\right)\tau'_{\nu_\phi}(1) \quad \mu_\phi.a.e$$

where ν_ϕ is the natural projection of μ_ϕ on $\sum_{r_1}$ and τ_{ν_ϕ} is given by

$$\tau_{\nu_\phi}(q) = \limsup_{n\to+\infty} \frac{1}{n \ln r_1} \ln \sum_{\omega\in\sum_{r_1}^n} \nu_\phi(I(\omega))^q$$

We mention that the family ξ of the almost-squares allows the calculation of the Hausdorff dimension (see [8]) when $\sum_r$ is endowed with the metric product, so we conclude the following result.

Corollary. *$\sum_A$ is endowed with the metric product. We have*

$$dim\left(\mu_\phi\right) = \frac{1}{\ln r_2} h\left(\mu_\phi\right) - \left(\frac{1}{\ln r_1} - \frac{1}{\ln r_2}\right)\tau'_{\nu_\phi}(1)$$