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Generalized two-qubit whole and half Hilbert–Schmidt separability probabilities

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A B S T R A C T

Compelling evidence – though yet no formal proof – has been adduced that the probability that a generic (standard) two-qubit state (ρ) is separable/disentangled is $\frac{8}{33}$ (arXiv:1301.6617, arXiv:1109.2560, arXiv:0704.3723). Proceeding in related analytical frameworks, using a further determinantal $4F3$-hypergeometric moment formula (Appendix A), we reach, *via* density-approximation (inverse) procedures, the conclusion that one-half ($\frac{4}{33}$) of this probability arises when the determinantal inequality $|\rho^{PT}| > |\rho|$, where PT denotes the partial transpose, is satisfied, and, the other half, when $|\rho| > |\rho^{PT}|$. These probabilities are taken with respect to the flat, Hilbert–Schmidt measure on the fifteen-dimensional convex set of 4×4 density matrices. We find fully parallel bisection/equipartition results for the previously adduced, as well, two-"re[al]bit" and two-"quater[nionic]bit" separability probabilities of $\frac{29}{64}$ and $\frac{26}{323}$, respectively. The new determinantal $4F3$-hypergeometric moment formula is, then, adjusted (Appendices B and C) to the boundary case of minimally degenerate states ($|\rho| = 0$), and its consistency manifested – also using density-approximation – with an important theorem of Szarek, Bengtsson and Życzkowski (arXiv:quant-ph/0509008). This theorem states that the Hilbert–Schmidt separability probabilities of generic minimally degenerate two-qubit states are (again) one-half those of the corresponding generic nondegenerate states.

1. Introduction

The problem of determining the probability that a bipartite/multipartite quantum state of a certain random nature exhibits a particular entanglement characteristic is clearly of intrinsic "philosophical, practical, physical" [1] interest [1–7]. We have reported [8,9] major advances, in this regard, with respect to the "separability/disentanglement probability" of generalized two-qubit states (representable by 4×4 density matrices ρ), endowed with the flat, Hilbert–Schmidt measure [5,10]. Most noteworthy, a concise formula [9, Eqs. (1)–(3)]

$$P(\alpha) = \sum_{i=0}^{\infty} f(\alpha + i), \tag{1}$$

where

$$f(\alpha) = P(\alpha) - P(\alpha+1) = \frac{q(\alpha) 2^{-4\alpha-6} \Gamma(3\alpha + \frac{5}{2}) \Gamma(5\alpha + 2)}{3\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(5\alpha + \frac{13}{2})}, \tag{2}$$

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and

$$q(\alpha) = 185000\alpha^5 + 779750\alpha^4 + 1289125\alpha^3 + 1042015\alpha^2 + 410694\alpha + 63000$$
$$= \alpha\left(5\alpha\left(25\alpha\left(2\alpha(740\alpha + 3119) + 10313\right) + 208403\right) + 410694\right) + 63000 \quad (3)$$

has emerged that yields for a given α, where α is a random-matrix-Dyson-like-index [11,12], the corresponding Hilbert–Schmidt separability probability $P(\alpha)$. The setting $\alpha = 1$ pertains to the fifteen-dimensional convex set of (standard/conventional, off-diagonal *complex*-entries) two-qubit density (4×4 Hermitian, unit-trace, positive-semidefinite) matrices.

The succinct formula yields (to arbitrarily high numerical precision) $P(1) = \frac{8}{33}$ (cf. [13], [14, Eq. B7], [15, sec. VII]). It is interesting to note that in this standard quantum-mechanical case [16], the probability seems of a somewhat simpler nature (smaller numerators and denominators) than the value $P(\frac{1}{2}) = \frac{29}{64}$ obtained for the ("attractive toy model" [4]) nine-dimensional convex set of 4×4 (two-"rebit") density matrices with *real* entries [17], or, the value $P(2) = \frac{26}{323}$ derived for the twenty-seven-dimensional convex set of 4×4 (two-"quaterbit" [18]) density matrices with *quaternionic* entries [19,20]. (Let us note that $P(\frac{3}{2}) = \frac{36061}{262144}$ [9, p. 9]. However, unlike the results for $\alpha = \frac{1}{2}$, 1 and 2, we have not been able to obtain this value through direct density-matrix calculations. This disparity may be attributable to the proposition that the only associative real division algebras are the real numbers, complex numbers, and quaternions [21].)

Fei and Joynt [22] have recently found strong support for these three primary conjectures by Monte Carlo sampling, using the extraordinarily large number of 5×10^{11} points for each of the three cases (cf. [23, Eq. (30)], [24]).

1.1. Multi-step derivation of concise formula

These simple rational-valued α-parameterized separability probabilities and the formula $P(\alpha)$ above that yields them were obtained through a number of distinct steps of analysis. First, based on extensive computations (employing Cholesky matrix decompositions/parameterizations, Dirichlet measure and integration over spheres), we inferred the (yet formally unproven) determinantal-moment formula [8, p. 30] (cf. [25, Eq. (28)], [26])

$$\left\langle \left|\rho^{PT}\right|^n \right\rangle = \frac{n!\,(\alpha+1)_n\,(2\alpha+1)_n}{2^{6n}\left(3\alpha+\frac{3}{2}\right)_n\left(6\alpha+\frac{5}{2}\right)_{2n}} + \frac{(-2n-1-5\alpha)_n\,(\alpha)_n\left(\alpha+\frac{1}{2}\right)_n}{2^{4n}\left(3\alpha+\frac{3}{2}\right)_n\left(6\alpha+\frac{5}{2}\right)_{2n}}\,{}_5F_4\left(\begin{matrix}-\frac{n-2}{2}, -\frac{n-1}{2}, -n, \alpha+1, 2\alpha+1\\ 1-n, n+2+5\alpha, 1-n-\alpha, \frac{1}{2}-n-\alpha\end{matrix}; 1\right).$$

The brackets here denote expectation with respect to Hilbert–Schmidt (Euclidean) measure, while $5F4$ indicates a particular generalized hypergeometric function. The partial transpose of ρ, obtainable by transposing in place its four 2×2 blocks, is denoted by ρ^{PT}.

The first 7501 of these moments ($n = 0, 1, \ldots, 7500$) were employed as input to a Mathematica program of Provost [27, pp. 19–20], implementing a Legendre-polynomial-based-density-approximation routine. From the high-precision, exact-arithmetic results obtained, we were able to formulate highly convincing, well-fitting conjectures (including the above-mentioned $\frac{8}{33}$ for $\alpha = 1$) as to underlying simple rational-valued separability probabilities. Then, with the use of the Mathematica FindSequenceFunction command applied to the sequence ($\alpha = 1, 2, \ldots, 32$) – or, fully equivalently, $\alpha = \frac{1}{2}, \ldots, \frac{63}{2}$ – of these conjectures, and simplifying manipulations of the lengthy Mathematica result generated, we derived a multi-term $7F6$ hypergeometric-based formula [9, Fig. 3] (cf. [28, Eq. (11)]), with argument $\frac{27}{64} = (\frac{3}{4})^3$, for the conjectured values. Then, Qing-Hu Hou (private communication) applied a highly celebrated ("creative telescoping") algorithm of Zeilberger [29] to this $7F6$-based expression to obtain the concise separability probability formula (1)–(3) for $P(\alpha)$ itself [9, Figs. 5, 6].

1.2. General remarks

Let us note that although the extensive symbolic and numeric computations conducted throughout this broad research project, have not furnished the rigorous proofs we, of course, strongly desire, they have been central to the testing of different approaches, and to the advancement of the specific determinantal-moment conjectures used for separability-probability evaluation. The conjectures take the form of equations asserted to hold for infinite ranges of parameter values, which can be verified for specific values of these parameters by symbolic computation.

Parallel programs to this one are being pursued in which: (1) the theoretically-important Bures (minimal monotone) measure [5,30,31] – rather than the Hilbert–Schmidt one – is applied to the 4×4 density matrices; and (2) the 6×6 (qubit–qutrit) systems are studied with the Hilbert–Schmidt measure appropriate to them. Considerably less progress has so far been achieved in these areas. No *general* moment formulas have yet been advanced, with explicit specific moment calculations having been implemented for the real and complex density matrices, so far for $\alpha = \frac{1}{2}$ and $\alpha = 1$ for small values of n [8, sec. 6], [32,33].