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A fundamental theorem for hypersurfaces in semi-Riemannian warped products

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ABSTRACT

We find necessary and sufficient conditions for nondegenerate arbitrary signature manifolds to be realized as hypersurfaces in a large class of warped products manifolds. As an application, we give conditions for a 3-dimensional hypersurface in a 4-dimensional Robertson–Walker spacetime to be foliated by surfaces with lightlike or zero mean curvature and hence describe a way to study horizons in such spacetimes.

1. Introduction

Given an immersed hypersurface in a spacetime, the Einstein equations satisfied on the ambient space impose relationships on the extrinsic and intrinsic curvatures, the so-called constraint equations. In this sense, Y. Choquet-Bruhat and R. Geroch obtained one of the main results in General Relativity, namely, if a Riemannian 3-manifold equipped with a given $(2, 0)$-tensor K satisfies the constraint equations, then it can be embedded as a Cauchy hypersurface in a unique maximal 4-dimensional globally hyperbolic Lorentzian manifold whose Riemann tensor is trace-free, in such a way that the given tensor K is its second fundamental form (see [1]). This is the so-called Cauchy problem.

In addition, this can be seen as constructing solutions to Einstein's equation, for which there are four well established techniques, namely

1. a spacelike Cauchy problem,
2. a boundary–initial value problem,
3. a characteristic Cauchy problem on two transverse hypersurfaces, or
4. a characteristic Cauchy problem on the light-cone.

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(See [2] and references therein.) These techniques can be regarded as particular cases of a more general, very interesting problem of Submanifold Theory, namely, to determine when a (semi-)Riemannian manifold (M^n, g) can be immersed into a (semi-)Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$. This problem is two-fold, depending on what you fix. On one hand, the target is to find in which ambient space can be immersed a specific manifold. This is the case of the above mentioned ones, like the Cauchy Problem. On the other hand, it is also possible to set the ambient space and try to find necessary and sufficient conditions for the existence of submanifolds with certain properties, known as *Fundamental Theorems*. In either case, the Gauß, Ricci and Codazzi equations must be satisfied by any submanifold, and their rule consists of relating the intrinsic and extrinsic curvatures. In fact, the classical Fundamental Theorem of Submanifolds states that they are necessary and sufficient conditions for a semi-Riemannian n-dimensional manifold to admit a (local) immersion into spaces of constant sectional curvature of dimension $n + d$. If the ambient space is not of constant sectional curvature, proving fundamental theorems is technically difficult and there are only few other results known by the authors. For instance, B. Daniel obtained in [3] a fundamental theorem for hypersurfaces in the Riemannian products $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, looking for tools to work with minimal surfaces in such manifolds when $n = 2$. J. Roth generalized B. Daniel's theorem to spacelike hypersurfaces in some Lorentzian products (see [4] and also [5,6] or [7] for related works). It is also worth pointing out that this geometric framework has been studied by discussing the causal character of the submanifold, namely timelike, lightlike, or spacelike. However, in a recent work, M. Mars showed a common formulation for all causal characters (see [8]). In addition, the classical moving frame technique by E. Cartan allows to construct immersions from a smooth manifold into a Lie group with associated Lie algebra $\mathfrak{g}$, if and only if, there exists a $\mathfrak{g}$-valuated 1-form satisfying the so-called Maurer–Cartan equation. Sometimes, the Maurer–Cartan equation holds due to the structure equations (see for instance [9,6]).

Now, we recall that the spaces of constant sectional curvature are examples of Einstein manifolds, i.e., manifolds for which the metric is a solution to the vacuum Einstein field equations (with cosmological constant). In order to get a deeper understanding of the global characteristics of the universe including its topology, a generalization of Einstein manifolds is needed, especially to the non-vacuum case. First generalizations are quasi-Einstein manifold. Examples of such manifolds include Robertson–Walker-spaces, which play an important role in standard models of cosmology and arise as solutions of the non-vacuum Einstein equations. Geometrically speaking, they are special cases of semi-Riemannian warped products.

Our main result in this paper is a *fundamental theorem* for non-degenerate hypersurfaces in a family of semi-Riemannian warped products $\varepsilon I \times_a \mathbb{M}^n_k(c)$, where $\varepsilon = \pm 1$, $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor and $\mathbb{M}^n_k(c)$ is the semi-Riemannian space form of index k and constant curvature $c \in \{1, 0, -1\}$. In this way, this allows to study existence problems of spacelike and timelike hypersurfaces in this family of warped products. As a particular case, we are describing a way to study 3-dimensional spacelike or timelike surfaces in 4-dimensional Robertson–Walker spacetimes. Thus, we are not following M. Mars' approach, and thus we are discarding the timelike case. Since our computations are made for any dimension, they might also allow to study surfaces in toy-models or supergravities, but this has to be explored.

For a hypersurface M in $\varepsilon I \times_a \mathbb{M}^n_k(c)$, the vector field ∂_t $(t \in I)$ decomposes in its tangent and normal parts, i.e., $\partial_t = T + \varepsilon_{n+1} T_{n+1} e_{n+1}$ where e_{n+1} is a (local) normal unit vector field, $\varepsilon_{n+1} = \pm 1$ shows its causal character and T_{n+1} is the corresponding coordinate. It turns out that the Gauss and Codazzi equations depend not only on the shape operator A, but also on the vector field T and on the warping function a, as well as on the function T_{n+1}. Recall that the Ricci equation provides no information for hypersurfaces (i.e., codimension 1). Moreover, in order to construct the immersion, a third equation is needed, namely the covariant derivative of T, *which cannot be obtained from Gauß and Codazzi equations*. Based on these three necessary conditions, we state in Definition 1 all needed tools. Theorem 1 states the necessary and sufficient conditions for the existence of a (local) metric immersion $\chi : \mathcal{U} \subset M \to \varepsilon I \times_a \mathbb{M}^n_k(c)$. Technically, we are using the previously mentioned technique of moving frames. As an application, in Corollaries 2 and 3, we obtain conditions for such a 3-dimensional hypersurface to be foliated by surfaces whose mean curvature vector is either lightlike or zero, in Robertson–Walker spacetimes. In this way, we are including surfaces with vanishing mean curvature vector, MOTS and mixed cases. This can be helpful for the study of horizons on Robertson–Walker spacetimes with spacelike or timelike causal character, including Marginally Outer Trapped Tubes.

2. Preliminaries

Let (P, g_P) be a semi-Riemannian manifold of dimension $\dim P = m$. All our manifolds will be connected and of class C^∞, unless otherwise stated. We consider a smooth function $a : I \subset \mathbb{R} \to \mathbb{R}^+$, a (sign) constant $\varepsilon = \pm 1$ and the warped product

$$\bar{P}^{m+1} = \varepsilon I \times_a P, \qquad \langle , \rangle = \varepsilon \mathrm{d}t^2 + a^2(t) g_P.$$

Clearly, the unit vector field $\partial/\partial t = \partial_t$ will play a crucial role on the manifold $\bar{P}^{n+1}$. We set our convention for the curvature operator $\mathcal{R}$ of a connection $\mathcal{D}$ as

$$\mathcal{R}(X, Y)Z = \mathcal{D}_X \mathcal{D}_Y Z - \mathcal{D}_Y \mathcal{D}_X Z - \mathcal{D}_{[X,Y]} Z.$$

Let $\bar{R}_P$ and R_P be the curvature operator of $\bar{P}^{n+1}$ and P, respectively. Since there is no confusion, we will use the same notation for the associated curvature tensors along the paper. Let $\mathbf{D}$ be the Levi-Civita connection of $\bar{P}^{n+1}$. We recall the following formulae, which can be checked in [10], bearing in mind the change of sign.