Journal of Geometry and Physics 90 (2015) 71–87

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

A Lagrangian for Hamiltonian vector fields on singular Poisson manifolds

Yahya Turki *

Université de Lorraine, Institut Elie Cartan de Lorraine UMR 7502, Metz, F-57045, France
Université de Monastir, Faculté des Sciences de Monastir, Avenue de l'Environnement 5019 Monastir, Tunisie

ARTICLE INFO

Article history:
Received 28 January 2014
Received in revised form 26 October 2014
Accepted 29 November 2014
Available online 8 December 2014

Keywords:
Differential geometry
Poisson geometry
Poisson sigma-models
Hamiltonian systems

ABSTRACT

On a manifold equipped with a bivector field, we introduce for every Hamiltonian a Lagrangian on paths valued in the cotangent space whose stationary points project onto Hamiltonian vector fields. We show that the remaining components of those stationary points tell whether the bivector field is Poisson or at least defines an integrable distribution—a class of bivector fields generalizing twisted Poisson structures that we study in detail.

1. Introduction

Poisson σ-models, developed by Ikeda [1] and Schaller–Strobl [2] is now a well-developed theory, well-known to give an alternative approach to Kontsevich star-product, see Cattaneo–Felder [3]. The Poisson sigma-models is based on the study of a certain functional, defined on vector bundle morphisms from the tangent space of a cylinder to the cotangent space of a manifold M equipped with a bivector field π. Our purpose is to study a very natural functional $\mathcal{L}^H$, which is in the same spirit as the one defining Poisson σ-models, but the dimension of the source manifold is 1 and M comes equipped with a bivector field and a function. Explicitly, this functional is given by:

$$\mathcal{L}^H(\alpha) = \int_0^1 \left\langle \mathcal{X}_H|_{x(t)} - \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \alpha(t) \right\rangle \mathrm{d}t, \tag{1}$$

with $\mathcal{X}_H$ being the Hamiltonian vector field. Our functional is also inspired by the celebrated functional of Weinstein [4], which is defined on paths valued in an exact symplectic manifold $(M, \omega = \mathrm{d}\beta)$:

$$\mathcal{L}^H(x) = \int_0^1 \left(\left\langle \beta_{x(t)}, \frac{\mathrm{d}x(t)}{\mathrm{d}t} \right\rangle - H(x(t)) \right) \mathrm{d}t. \tag{2}$$

See Section 4 for a more precise relation between all those functionals. The initial purpose of the present article is to state results of the form "A bivector field is of type X if and only if the stationary points of $\mathcal{L}^H$ are, for all H, of type Y". We were especially interested to find the Y corresponding to X = "Poisson" and the X corresponding to Y = "cotangent paths". To be able to state such a result, we are obliged to introduce several new notions.

* Correspondence to: Université de Lorraine, Institut Elie Cartan de Lorraine UMR 7502, Metz, F-57045, France.
E-mail address: yahya.turki@univ-lorraine.fr.

http://dx.doi.org/10.1016/j.geomphys.2014.11.015

1. Quasi-cotangent paths (with respect to a function $H \in C^\infty(M)$), which are simply paths $\alpha(t)$ in T^*M for which $\pi^\#(\alpha) - \frac{dx(t)}{dt}$ is $\mathcal{X}_H$-invariant, with x the base path of M. In particular cotangent paths are quasi-cotangent.
2. Foliated bivector fields, which are defined to be those for which the distribution $\pi^\#(T^*M)$ is integrable. This is a subtle notion that we claim to have an interest of its own. For every singular integrable distribution in the sense of Sussmann [5,6], it is natural to look for an algebroid whose image through the anchor map gives the distribution, at least locally [7]. For foliated bivector fields, this algebroid is not a priori given and does not seem easy to guess. This open question will be addressed in a subsequent paper.
3. Weakly foliated bivector fields, which are defined to be those for which the distribution $\pi^\#(T^*M)$ is integrable, so to say, at every point, i.e. such that $\left[\pi^\#(T^*M), \pi^\#(T^*M)\right]\big|_m$ is the image of $\pi^\#_m$ at every point m of the manifold M.

Our main result states the following, see Theorem 26 for more details. Let M be a manifold and π be a bivector field on M.

1. If X = "foliated" then Y = "cotangent paths" for every function $H \in C^\infty(M)$.
2. If X = "weakly foliated" then Y = "cotangent paths" for every function $H \in C^\infty(M)$.
3. X = "Poisson" if and only if Y = "quasi-cotangent paths" for every function $H \in C^\infty(M)$.

The reader may notice that we do not work with loops, but with paths, which is highly surprising since from Weinstein's Lagrangian (2), it is periodicity that makes things interesting. We are in fact also interested by the periodic case, but this should be postponed to a subsequent article.

The paper is organized as follows. Section 2 is devoted to the study of a new type of bivector fields that we call foliated bivector fields, which are defined to be those bivector field π for which the distribution $\pi^\#(T^*M)$ is integrable. As will appear in the course of Section 2, foliated bivector fields are strongly related to twisted Poisson structures [8,9]. Indeed, these structures, also called Poisson structures with background, are shown in Proposition 3 to be foliated. Also, every foliated bivector field comes from a twisted Poisson structure at regular points, see Proposition 4. Last, each leaf of a foliated Poisson structure comes equipped with a twisted Poisson structure of maximal rank, see Theorem 9. Conformally Poisson structures are also among examples (see Proposition 6), indeed, in contrast with Poisson structures, foliated bivector fields are a $C^\infty(M)$-module. Also, weak Poisson structures are defined and related to their "strong" counterpart.

In Section 3, the Lagrangian briefly introduced in (1) is studied in details, and the main result of this paper, Theorem 26, is stated and proved. Section 4 explains the relation with Poisson σ-models.

I express my gratitude to the University of Monastir for two "bourses d'alternance" that I received while preparing this manuscript. I also thank Marco Zambon for useful comments on the first version of the manuscript.

2. Foliated bivector fields

2.1. Definitions and notations

We define foliated bivector fields and indicate here some definitions, notations and results which will be needed in the sequel. From M is an arbitrary manifold of dimension n, and $A^\bullet(M) := \sum_{k=0}^n A^k(M)$ is the Gerstenhaber algebra of multivector fields, equipped with the wedge product and the Schouten–Nijenhuis bracket, that we simply denote by $[\cdot\,,\cdot]$. Here, given a vector space E, for all $\pi \in \wedge^2 E$ we denote by $\pi^\#$ the morphism of vector bundle of $E^* \to E$ given by for all $\xi, \eta \in E^*$:

$$\left\langle \pi^\#(\xi), \eta \right\rangle = \langle \pi, \xi \wedge \eta \rangle \tag{3}$$

where $\langle \alpha \wedge \beta, \xi \wedge \eta \rangle = \langle \alpha, \xi \rangle \langle \beta, \eta \rangle - \langle \alpha, \eta \rangle \langle \beta, \xi \rangle$ is the natural pairing between $\wedge^2 E$ and $\wedge^2 E^*$. We recall [10–12] that a *Poisson structure* is a bivector field $\pi \in A^2(M)$ satisfying $[\pi, \pi] = 0$. This property implies that the biderivation of $\mathbf{C}^\infty(M)$ defined by

$$\{F, G\} := \langle \pi, dF \wedge dG \rangle$$

is a Lie bracket, and that the map assigning to a function F its Hamiltonian vector field $\mathcal{X}_F := \pi^\#(dF)$ is an anti-Lie algebra morphism, i.e. $[\mathcal{X}_F, \mathcal{X}_G] = \mathcal{X}_{\{F,G\}}$ for all $F, G \in \mathbf{C}^\infty(M)$. This implies that:

$$\left[\pi^\#(\Omega^1(M)), \pi^\#(\Omega^1(M))\right] \subset \pi^\#(\Omega^1(M)). \tag{4}$$

Eq. (4) means that the locally finitely generated $\mathbf{C}^\infty(M)$-module $\pi^\#(\Omega^1(M))$ (which is a sub-module of $A^1(M)$) is a *integrable distribution*. However, a bivector π can verify the latter property if it is not Poisson. This is precisely the point of the following definition:

Definition 1. Let M be a manifold. We call *foliated bivector field* a bivector field $\pi \in A^2(M)$ such that the image of $\pi^\# : \Omega^1(M) \to A^1(M)$ is a integrable distribution.

The following lemma is immediate.

Lemma 2. *A bivector field $\pi \in A^2(M)$ is foliated if and only if for any pair F, G of functions on M, there exists a 1-form $\alpha_{F,G}$ such that $[\mathcal{X}_F, \mathcal{X}_G] = \pi^\#(\alpha_{F,G})$.*