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Invariant variational problems on homogeneous spaces

Cornelia Vizman

Department of Mathematics, West University of Timişoara, Bd. V. Pârvan 4, 300223-Timişoara, Romania

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ABSTRACT

Covariant Euler–Poincaré equations on homogeneous spaces are studied, including the special case of strands on homogeneous spaces. Space–time strands on homogeneous spaces are treated also dynamically, using affine Euler–Poincaré reduction.

1. Introduction

Classical Euler–Poincaré equations on the Lie algebra $\mathfrak{g}$ of the Lie group G arise via reduction of the variational principle for a G-invariant Lagrangian $L : TG \to \mathbb{R}$: with a restricted class of variations, the extremals of the integral of the reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ correspond to extremals of the original variational problem for L [1].

The variational reduction program is extended in [2] to the setting of a principal G-bundle P over M, see also [3]. The G-invariant Lagrangian density is defined on J^1P, the first jet bundle of sections of P. Its quotient space J^1P/G is identified with the bundle of principal connections on P. The reduced equations that are obtained can be seen as generalized EP equations for field theory. Zero curvature conditions on the reduced solutions are imposed for the existence of the original solutions.

The case when the manifold M is sliced leads to the reduced equations for space–time strands, also called sliced covariant EP equations [4,5]. Special cases are the G-strands [6]. When M is sliced, the covariant EP reduction can be reformulated as a classical dynamic reduction. Two approaches have been proposed in [4] and in [3]. We shall follow the approach of [3] based on affine EP reduction. See also [5] for more explanation of the link between G-strands, covariant EP, and affine EP reductions.

Reduction theory for the principal fiber bundle P by a subgroup $H \subset G$ of symmetries is performed in [7]. The configuration bundle of the reduced problem is now J^1P/H. A fixed connection on the principal bundle $P \to P/H$ is used for splitting the reduced equation in two equations, the first of them containing the Euler–Lagrange operator applied to sections of the fiber bundle P/H over M (maps from M to G/H in the trivial bundle case).

In this paper we focus on reduced variational problems on homogeneous spaces G/H for a more restrictive class of Lagrangians. Because of these stronger invariance requirements the second equation in the splitting mentioned above

E-mail address: vizman@math.uvt.ro.

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vanishes identically. That is why our approach does not require to fix a connection. We emphasize the special invariance properties of the reduced equations that come up after lifting the Lagrangian to the Lie group G as in [8]. We pass from the classical case of curves to strands and then to more general maps, especially space–time strands, all of them taking values in homogeneous spaces.

Reduced equations for G-invariant parameter dependent Lagrangians, called Euler–Poincaré equations for symmetry breaking, are introduced in [9]. Reduction can be performed for parameter dependent Lagrangians on homogeneous spaces too [10]. We use the same approach to adapt to homogeneous spaces a result from [3], where space–time strands on Lie groups are treated dynamically. To an invariant field theoretic Lagrangian density on the first jet bundle of a principal bundle one associates a gauge group invariant dynamic Lagrangian on the tangent bundle of the gauge group depending parametrically on a connection. Covariant and dynamic reduction lead to equivalent reduced equations. We treat only the trivial principal bundle case.

The plan of the paper is the following. In Section 2 we recall the Euler–Poincaré equations on homogeneous spaces [8] and the parameter dependent version which leads to Euler–Poincaré equations for symmetry breaking [10]. In Section 3 we treat covariant Euler–Poincaré equations for maps into homogeneous spaces. In the last section we compare covariant reduction for space–time strands in homogeneous spaces with dynamic reduction, as done in [3] for Lie groups.

2. Euler–Poincaré equations on homogeneous spaces

In this section we recall the Euler–Poincaré equations on homogeneous spaces [8] and the parameter dependent version which leads to Euler–Poincaré equations for symmetry breaking [10].

2.1. Classical Euler–Poincaré equations

The Euler–Lagrange (EL) equations for a right G-invariant Lagrangian $L : TG \to \mathbb{R}$, written for the reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of G, are the Euler–Poincaré (EP) equations [1]

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} = 0. \tag{2.1}$$

Here ξ is a curve in $\mathfrak{g}$, the right logarithmic derivative of a curve g in G, i.e. $\xi = \dot g g^{-1}$, and $\delta\ell/\delta\xi$ denotes the functional derivative: $\left(\frac{\delta\ell}{\delta\xi}, \zeta\right) = \frac{d}{dt}\Big|_{t=0}\ell(\xi + t\zeta)$ for all $\zeta \in \mathfrak{g}$.

Logarithmic derivative and homogeneous spaces. The right logarithmic derivative for curves in a homogeneous space G/H can be seen as a multivalued map by considering the logarithmic derivatives of all the lifted curves g in G of a given curve $\bar g$ in G/H. All of them belong to the same orbit of the left action of $C^\infty(I, H)$ on $C^\infty(I, \mathfrak{g})$:

$$h \cdot \xi = \mathrm{Ad}_h \xi + \delta^r h, \tag{2.2}$$

because $\delta^r(hg) = \mathrm{Ad}_h \delta^r g + \delta^r h$. Thus we can define [8]

$$\bar\delta^r : C^\infty(I, G/H) \to C^\infty(I, \mathfrak{g})/C^\infty(I, H), \qquad \bar\delta^r \bar g = C^\infty(I, H) \cdot \delta^r g. \tag{2.3}$$

Similarly, a right logarithmic derivative for homogeneous space valued maps on a smooth manifold M can be given by

$$\bar\delta^r : C^\infty(M, G/H) \to \Omega^1(M, \mathfrak{g})/C^\infty(M, H), \qquad \bar\delta^r \bar g = C^\infty(M, H) \cdot \delta^r g, \tag{2.4}$$

where $\delta^r g = dgg^{-1}$ is the right logarithmic derivative of g and the action of $C^\infty(M, H)$ on $\Omega^1(M, \mathfrak{g})$ is the gauge action

$$h \cdot \sigma = \mathrm{Ad}_h \sigma + \delta^r h, \tag{2.5}$$

restriction of the action of the gauge group $C^\infty(M, G)$ on the space of connections for the trivial principal G-bundle over M.

The tangent bundle TG carries a natural group multiplication. Given a Lie subgroup H of G, its tangent bundle TH is a Lie subgroup of TG and there is a canonical diffeomorphism between TG/TH and $T(G/H)$. The following are equivalent data: a right G-invariant Lagrangian $\bar L$ on $T(G/H)$, a left TH-invariant and right G-invariant Lagrangian L on TG, as well as an $\mathfrak{h}$-invariant and $\mathrm{Ad}(H)$-invariant reduced Lagrangian ℓ on $\mathfrak{g}$.

Proposition 2.1 ([8]). *Given a reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ that satisfies $\ell(\mathrm{Ad}_h\xi + \eta) = \ell(\xi)$ for all $h \in H$ and $\eta \in \mathfrak{h}$, the EP equations*

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} = 0 \tag{2.6}$$

are $C^\infty(I, H)$-invariant for the action (2.2).

In other words (2.6) being an equation for $C^\infty(I, H)$-orbits in $C^\infty(I, \mathfrak{g})$, is an equation for the multivalued right logarithmic derivative for curves in the homogeneous space (2.3).

A special EL equation is the geodesic equation for a right G-invariant Riemannian metric on G/H, i.e. Euler equation on homogeneous spaces [11]. Examples are the Hunter–Saxton equation and its multidimensional version, geodesic equations on $\mathrm{Diff}(S^1)/S^1$ and $\mathrm{Diff}(M)/\mathrm{Diff}_{\mathrm{vol}}(M)$.