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Complex structures adapted to magnetic flows

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ABSTRACT

Let M be a compact real-analytic manifold, equipped with a real-analytic Riemannian metric g, and let β be a closed real-analytic 2-form on M, interpreted as a magnetic field. Consider the Hamiltonian flow on T^*M that describes a charged particle moving in the magnetic field β. Following an idea of T. Thiemann, we construct a complex structure on a tube inside T^*M by pushing forward the vertical polarization by the Hamiltonian flow "evaluated at time i". This complex structure fits together with $\omega - \pi^*\beta$ to give a Kähler structure on a tube inside T^*M. When $\beta = 0$, our magnetic complex structure is the adapted complex structure of Lempert–Szőke and Guillemin–Stenzel.

We describe the magnetic complex structure in terms of its $(1, 0)$-tangent bundle, at the level of holomorphic functions, and via a construction using the embeddings of Whitney–Bruhat and Grauert. We describe an antiholomorphic intertwiner between this complex structure and the complex structure induced by $-\beta$, and we give two formulas for local Kähler potentials, which depend on a local choice of vector potential 1-form for β. Finally, we compute the magnetic complex structure explicitly for constant magnetic fields on $\mathbb{R}^2$ and S^2.

1. Introduction

1.1. Adapted complex structures

Adapted complex structures were introduced, independently and in different but equivalent ways, by L. Lempert and R. Szőke [1,2] and by V. Guillemin and M. Stenzel [3,4]. Let (M, g) be a real-analytic Riemannian manifold, let TM be the tangent bundle of M, and let T^RM denote the "tube" of radius R, that is, the set of vectors in TM with length less than $R > 0$. For each unit-speed geodesic γ in M, we can define a map Ψ_γ of the complex plane into TM by setting

$$\Psi_\gamma(\sigma + i\tau) = N_\tau \dot{\gamma}(\sigma) \in T_{\gamma(\sigma)}M,$$

where N_τ is scaling in the fibers by τ. In the terminology of Lempert–Szőke [1, Def 4.1], a complex structure on some T^RM is *adapted* (to the metric on M) if, for each γ, the map Ψ_γ is holomorphic as a map of the strip $\{\sigma + i\tau : |\tau| < R\} \subset \mathbb{C}$ into T^RM. If M is compact, there is a unique adapted complex structure on T^RM for all sufficiently small R. Guillemin and Stenzel defined complex structures on the cotangent bundle T^*M by means of a Kähler potential and an involution. Their approach turns out to be equivalent to that of Lempert–Szőke after identification of the tangent and cotangent bundles.

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For certain very special manifolds M, the adapted complex structure exists globally, that is, on all of $TM \cong T^*M$. Examples of such manifolds include compact Lie groups with bi-invariant metrics, compact symmetric spaces, and the Gromoll–Meyer exotic 7-sphere [5]. When M is a compact Lie group with a bi-invariant metric, nice results hold for the geometric quantization of T^*M with the polarization coming from the adapted complex structure [6–8].

1.2. Thiemann's method

Meanwhile, in [9, Sec 2.1], T. Thiemann proposes a "complexifier" method for introducing complex structures on cotangent bundles of manifolds. Let C be a smooth function on T^*M and let X_C be the associated Hamiltonian vector field. Let f be a function that is constant along the leaves of the cotangent bundle and define

$$f_{\mathbb{C}} = e^{iX_C}(f), \tag{1.1}$$

provided that this can be defined in some natural way either on all of T^*M or on some portion thereof. (For real σ, $\exp(\sigma X_C)f$ is just the composition of f with the classical flow generated by C. To put $\sigma = i$, we need to analytically continue the expression $\exp(\sigma X_C)f$ with respect to σ.) Thiemann proposes that the complex structure associated to the function C is the one for which the holomorphic functions are precisely the functions of the form $f_{\mathbb{C}}$, where f is constant along the fibers of T^*M.

For a general C (even assumed to be real analytic), it is not clear to what extent one can carry out this program, because of convergence questions associated to the analytic continuation. Nevertheless, in the case that M is a compact Lie group with bi-invariant metric, if we take C to be half the length-squared in the fibers, then it is not hard to show that Thiemann's prescription makes sense and gives the adapted complex structure on T^*M. (See Equation (3.37) in [10], Equation (3.8) in [11], and Section 4 of [6].)

In [12], we analyze adapted complex structures from the point of view of Thiemann's complexifier method. We consider a compact Riemannian manifold and we take the complexifier to be the energy function E, equal to half the length-squared in the fibers, so that the flow of X_E is the geodesic flow. We give several different but equivalent ways of making sense of the analytic continuation in (1.1). We then show that the resulting complex structure is the adapted complex structure. From this point of view, we are able to give simple arguments for the known properties of the adapted complex structure, including the Kähler potential and involution of [3].

1.3. Magnetic complex structures

In the present paper, we apply Thiemann's method to construct a new family of complex structures on (co)tangent bundles of Riemannian manifolds, generalizing the adapted complex structure. Specifically, we consider a real-analytic Riemannian manifold together with a closed real-analytic 2-form β on M, where β is interpreted as a magnetic field. The dynamics of a charged particle moving in this magnetic field may be described as the Hamiltonian flow Φ_σ of the energy function E with respect to the "twisted" symplectic form $\omega^\beta := \omega - \pi^*\beta$, where ω is the canonical 2-form on T^*M and $\pi : T^*M \to M$ is the projection onto the base.

Following our approach in [12], we define a family of subspaces $P_z(\sigma)$ by pushing forward the (complexified) vertical subspace by Φ_σ. We show that there is some $R > 0$ for which the following hold. First, for all z in a tube $T^{*,R}M$, the map $\sigma \mapsto P_z(\sigma)$ has an analytic continuation to a disk of radius greater than one. Second, the subspaces $P_z(i)$ are the $(1,0)$-subspaces for an integrable almost complex structure on $T^{*,R}M$. Third, this complex structure fits together with the symplectic form $\omega - \pi^*\beta$ to give a Kähler structure on $T^{*,R}M$. In addition, we give a local expression for a Kähler potential in terms of a locally defined 1-form A on M with $dA = \beta$. In contrast to the case of adapted complex structures, inversion in the fibers (the map sending each $p \in T_x^*M$ to $-p$) is not antiholomorphic, but rather antiholomorphically intertwines the complex structures associated to β and $-\beta$.

In fact, $P_z(\sigma + i\tau)$ induces a Kähler structure on $T^{*,R}M$ for any $\sigma + i\tau \in D_{1+\varepsilon}$ with $\tau > 0$, both in the magnetic and $\beta = 0$ cases. In the $\beta = 0$ case, the recent papers [13], [14] give an alternate construction of this family of complex structures and study the induced family of Kähler quantizations of $T^{*,R}M$.

Although our main theorems are proved for the case where M is compact, the definitions also make sense for noncompact M. Similar results should hold on some neighborhood of the zero-section, but such a neighborhood in the noncompact case does not necessarily contain a tube around the zero-section.

We also compute the magnetic complex structure for the cases of a constant magnetic field on the plane $\mathbb{R}^2$ and the sphere S^2. In these cases, the complex structure can be computed explicitly and exists on the whole cotangent bundle. In the plane, we also compute explicitly a Kähler potential, and explain how it is related to the work of Krötz–Thangavelu–Xu [15] on heat kernel analysis on Heisenberg groups.

1.4. Geodesics in the space of Kähler structures

We conclude this introduction by briefly explaining the relationship between our results and geodesics in the space of Kähler structures. Details of this connection will appear in a subsequent work. The usual context for the study of these geodesics is a compact symplectic manifold N, equipped with a Kähler structure (J_0, ω_0). One then studies the space of all