Physica D 263 (2013) 57–73

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Twist number and order properties of periodic orbits

Emilia Petrisor *

Department of Mathematics, Polytechnic University of Timisoara, Pta Victoriei 2, 300006 Timisoara, Romania

HIGHLIGHTS

- Framework to study the twist number.
- Orbits of large twist numbers are born through period doubling bifurcation.
- Algorithm to compute the twist number.
- First examples of orbits of large twist number for standard-like maps.
- Point out relationship twist number-order properties of periodic orbits.

ARTICLE INFO

Article history:
Received 24 July 2012
Received in revised form
29 July 2013
Accepted 1 August 2013
Available online 24 August 2013
Communicated by T. Sauer

Keywords:
Twist map
Periodic orbit
Twist number
Bifurcation
Unordered orbit
1-cone function

ABSTRACT

A less studied numerical characteristic of periodic orbits of area preserving twist maps of the annulus is the twist or torsion number, called initially the amount of rotation Mather (1984) [2]. It measures the average rotation of tangent vectors under the action of the derivative of the map along that orbit, and characterizes the degree of complexity of the dynamics.

The aim of this paper is to give new insights into the definition and properties of the twist number and to relate its range to the order properties of periodic orbits. We derive an algorithm to deduce the exact value or a demi-unit interval containing the exact value of the twist number.

We prove that at a period-doubling bifurcation threshold of a mini-maximizing periodic orbit, the new born doubly periodic orbit has the absolute twist number larger than the absolute twist of the original orbit after bifurcation. We give examples of periodic orbits having large absolute twist number, that are badly ordered, and illustrate how characterization of these orbits only by their residue can lead to incorrect results.

In connection to the study of the twist number of periodic orbits of standard-like maps we introduce a new tool, called 1-cone function. We prove that the location of minima of this function with respect to the vertical symmetry lines of a standard-like map encodes a valuable information on the symmetric periodic orbits and their twist number.

1. Introduction

The study of the dynamics of area preserving positive twist maps of the annulus is mainly concerned with the characterization of its invariant sets, and the dynamical behavior of a map restricted to such sets. Among invariant sets, rotational periodic orbits have been thoroughly studied and classified according to their linear stability, extremal type, order properties (see [1] for a survey). A numerical characteristic associated with a periodic orbit is the rotation number, which measures the average rotation of the orbit around the annulus. In [2] Mather defined also the amount of rotation, which is called twist number in [3] or torsion number in [4]. The twist number of a periodic orbit characterizes the average rotation of tangent vectors under the action of the tangent map along the map's orbit. Mather related the twist number with the Morse index of the corresponding critical sequence (configuration), and Angenent [3] proved that in the space of (p, q)-sequences a critical point of the W_{pq}-action, corresponding to a periodic orbit of twist number greater than 0 is connected by the negative gradient flow of the action, through a heteroclinic connection, with a sequence corresponding to an orbit of zero twist number. Using topological arguments, Crovisier [4] proved the existence of orbits of zero-torsion (twist) number for any real number in the rotation number set of a twist map of the annulus (not necessarily area preserving).

In order to detect new classes of dynamical systems exhibiting periodic orbits of non-zero twist (torsion), Béguin and Boubaker gave [5] conditions ensuring that some area preserving diffeomorphisms of the disk $\mathbb{D}^2$, and particular diffeomorphisms of the torus $\mathbb{T}^2$ exhibit such orbits.

* Tel.: +40 729761332.
E-mail addresses: emilia.petrisor@gmail.com, emilia.petrisor@mat.upt.ro.

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http://dx.doi.org/10.1016/j.physd.2013.08.001

In Mather's and Angenent's definitions of the twist number this a positive quantity. However its natural definition leads to a signed number [4]. The twist number of periodic orbits of positive twist maps is zero or negative, while for those of negative twist maps it is zero or positive. In our approach we keep its natural sign and call absolute twist, the absolute value of the twist number.

During almost 30 years since the definition of the twist number was given, no periodic orbit of absolute twist number greater than 1/2 was detected in the dynamics of classical twist maps (standard map, Fermi–Ulam map, etc.). That is why we wonder under what conditions can such orbits appear.

From Aubry–Mather theory [6,7] it is known that non-degenerate minimizing and associated mini-maximizing (p, q)-type periodic orbits are well ordered and the minimizing orbit has zero twist number. Thus it is natural to investigate whether (p, q)-periodic orbits that are not given by this theory can be well-ordered or not and how order properties are related to the value of their twist number.

Our starting point was not however the study of the twist number. In an attempt to characterize the dynamical behavior of twist maps after the breakdown of the last KAM invariant torus (the case of standard map) or in a half annulus where no invariant circle exist (the case of Fermi–Ulam map [8], or tokamap [9]), we identified in the phase space of such maps, special regions where no minimizing periodic orbits can land. Instead we noticed that any periodic orbit having at least two points in such a region is badly ordered and typically has absolute twist greater than 1/2.

Thus we were led to a deeper study of the twist number of periodic orbits, and their properties. In this paper we complement the results on twist number reported in [2,3] and show that periodic orbits of non-zero twist numbers are typically born through a period doubling bifurcation. We give examples of periodic orbits that have large absolute twist number. As far as we know no such orbits were identified before in the study of the most known twist maps, standard-like maps. Some of these orbits are ordered and other are unordered. In order to explain why some twist maps can exhibit a sequence of bifurcations of a periodic orbit leading to increasing the absolute twist number up to its maximal value we introduce two subclasses of standard-like maps, USF maps and TSF maps.

The paper is organized as follows. In Section 2 we give the basic properties of twist area preserving diffeomorphisms of the annulus, relevant to our study. In order to derive in Section 4 sufficient conditions that favor the existence of ordered periodic orbits of large absolute twist number, we set up in Section 3 a framework of study, defining the class of twist maps exhibiting the so called strong folding property, and a function, called 1-cone function.

The 1-cone function is defined on the phase space of a twist map and takes negative values within the region where the map exhibits strong folding property. We prove that the restriction of this function to a periodic orbit gives information on the eigenvalues of the Hessian matrix associated with that orbit (Lemma 3.1). Analyzing the 1-cone function associated with a standard-like map, we show that such maps can have either a connected strong folding region including one of the vertical symmetry lines or a two-component strong folding region including both vertical symmetry lines. These results will be exploited in Section 5 to explain the variation of the twist number.

In Section 4 we revisit the definition and properties of the twist number of a periodic orbit based on the structure of the universal covering group of the group $SL(2, \mathbb{R})$ (a system of coordinates on this group allowing to decipher its topology is presented in Appendix B). The twist number is defined as the translation number of a circle map induced by the monodromy matrix associated with the periodic orbit. We review in Appendix A the properties of the translation number of an orientation preserving homeomorphism of the unit circle and point out the particularities of the translation number of circle homeomorphisms induced by matrices in $SL(2, \mathbb{R})$.

Theorem 4.2 gives the relationship between the twist number value of a (p, q)-periodic orbit, and the position of the real number 0 with respect to the sequence of interlaced eigenvalues of the Hessian matrix H_q, associated with the corresponding (p, q)-sequence, and of a symmetric matrix derived from H_q. This theorem complements results from [2,3].

Based on this theorem we derive an algorithm to deduce the exact value of the twist number or a demi-unit interval that contains the twist number.

The main result in Section 4 is the Proposition 4.1, which shows that periodic orbits of large absolute twist number are born through a period doubling bifurcation. More precisely it states that if at some threshold, a period doubling bifurcation of a mini-maximizing (p, q)-periodic orbit occurs, with transition from elliptic to inverse hyperbolic orbit, then the new born $2q$-periodic orbit is elliptic, having the twist number within the interval $(-1, -1/2)$.

In Section 5 we give examples of periodic orbits that have large absolute twist number and are unordered. A natural question is whether a positive twist map can also exhibit ordered (p, q)-periodic orbits (p, q relative prime integers), of twist number less than $-1/2$ (we note that periodic orbits of large absolute twist number, born through a period doubling bifurcation are of type $(2p, 2q)$). We give a positive answer to this question, giving an example of three-harmonic standard map having an ordered $(1, 2)$-periodic orbit, of twist number $\tau = -1$.

In order to illustrate the different range of the twist number of a periodic orbit in the three-harmonic standard map in comparison to that of a periodic orbit of the same type of the standard map, we deduce (Proposition 5.1) the bifurcations that a periodic orbit of a family of standard-like maps, with a uni-component strong folding region, undergo necessarily.

The concrete examples of periodic orbits of large absolute twist number, given in this section, illustrate that the classical method used for more than 30 years in classification of periodic orbits can lead to an erroneous conclusion.

2. Background on twist maps

We recall basic properties of twist maps, relevant to our approach. For a detailed presentation of classical results concerning dynamics of this type of maps, the reader is referred to [10,1].

Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle, $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$, the infinite annulus, and $\pi : \mathbb{R}^2 \to \mathbb{A}$, the covering projection, $\pi(x, y) = (x \bmod 1, y)$.

We consider a C^1-diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2, F(x, y) = (x', y')$, satisfying the following properties:

(i) F is exact area preserving map, isotopic to the identity;
(ii) $F(x + 1, y) = F(x, y) + (1, 0)$, for all $(x, y) \in \mathbb{R}^2$;
(iii) F has uniform positive twist property, i.e. $\frac{\partial F_1}{\partial y} \geq c > 0$.

The map F defines a C^1-diffeomorphism, $f : \mathbb{A} \to \mathbb{A}$, such that $\pi \circ F = f \circ \pi$. Both maps f and F are called *area preserving positive twist maps* or simply twist maps. F is a lift of f.

In the sequel we will switch from f to F, and conversely, without comment.

The motion of a point around the annulus is characterized by its rotation number. The orbit of a point $z \in \mathbb{A}$ has a rotation number if there exists the limit:

$$\rho = \lim_{n\to\infty} \frac{x_n - x}{n}, \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ is a lift of z, and $(x_n, y_n) = F^n(x, y)$. The rotation number does not depend on the chosen point z on the orbit or the lift (x, y). For different lifts of the map f, the corresponding rotation numbers differ by an integer.

Let p, q be two relative prime integers, $q > 0$. A (p, q)-type orbit of the twist map, F, is an orbit $(x_n, y_n) = F^n(x_0, y_0)$, $n \in \mathbb{Z}$, such