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Ordinary differential equations described by their Lie symmetry algebra[☆]

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ABSTRACT

The theory of Lie remarkable equations, *i.e.*, differential equations characterized by their Lie point symmetries, is reviewed and applied to ordinary differential equations. In particular, we consider some relevant Lie algebras of vector fields on $\mathbb{R}^k$ and characterize Lie remarkable equations admitted by the considered Lie algebras.

1. Introduction

In the context of the geometric theory of symmetries of (systems of) differential equations (DEs) [1–5], a natural problem is to see when a DE, either partial (PDE) or ordinary (ODE), is uniquely determined by its Lie algebra of point symmetries. The core of this paper is to investigate the inverse problem in the context of ODEs: given a Lie algebra $\mathfrak{s}$ of vector fields, how to construct ODEs having $\mathfrak{s}$ as a Lie point symmetry subalgebra and satisfying some specific properties, which will be clarified below, that ensure the uniqueness of such DE. The idea of describing DEs admitting a given Lie algebra of symmetries dates back at least to S. Lie, who stated that $u_{xx} = 0$ is the unique scalar 2nd order ODE, up to point transformations, admitting an 8-dimensional Lie algebra of symmetries. Of course, a similar idea also applies to PDE: for instance, in [6] (see also [7,8]), the author proved that the only scalar 2nd order PDE, with an unknown function and two independent variables, admitting the Lie algebra of projective vector fields of $\mathbb{R}^3$ as Lie point symmetry subalgebra is the Monge–Ampère equation $u_{xx}u_{yy}-u_{xy}^2 = 0$. The above idea plays a central role also in gauge theories, where one wants to obtain information on differential operators possessing a prescribed algebra of symmetries. The results of [9] go in this direction: 2nd order field equations possessing translational and gauge symmetries and the corresponding conservation laws (via Noether theorem) are always derivable from a variational principle.

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The standard procedure (also used in [6]) for obtaining a scalar DE admitting a prescribed Lie algebra of symmetries is that of computing the differential invariants of its prolonged action, under some regularity hypotheses; the invariant DE is then described by the vanishing of an arbitrary function of such invariants. If the prolonged action is not regular, invariant DEs can be obtained by a careful study of the singular set of the aforementioned action. An efficient method for obtaining invariant scalar ODEs in the latter case is that of using Lie determinants [10], which we shall employ for our purposes. See [11–14] for more approaches to the problem. In general, DEs do not possess a sufficient number of independent Lie point symmetries able to characterize them (among the others we recall KdV equation, Burgers' equation, Kepler's equations). In this case, one can ask if they can be characterized by a more general algebra of symmetries. A possible generalization of the concept of Lie remarkable equations is that suggested in [13,6]: this amounts to extending the category of symmetries used in the definitions of Lie remarkable equations to contact symmetries. For instance, the minimal surface equation of $\mathbb{R}^3$ is completely determined by its contact symmetry algebra [6]. Also, an example of high-order Lie remarkable equation in this 'extended' sense is

$$10u_{(3)}^3u_{(7)} - 70u_{(3)}^2u_{(4)}u_{(6)} - 49u_{(3)}^2u_{(5)}^2 + 280u_{(3)}u_{(4)}^2u_{(5)} - 175u_{(4)}^4 = 0, \tag{1}$$

where $u_{(k)} = d^ku/dx^k$, which possesses a 10-dimensional Lie algebra of contact symmetries (see [15,10]). Sometimes, in order to completely characterize a given DE, one should also consider non-local symmetries. This is the situation discussed in [16], where the idea of complete symmetry group was proposed and exploited in order to characterize uniquely Kepler's equation. This idea was subsequently exploited by several authors in different ways for characterizing many differential equations [17–23].

Following the terminology introduced in [24–28], we call *Lie remarkable* a DE which is completely characterized by its Lie algebra of point symmetries. Of course, this concept needs some cares and comments, which we will give below. Thus, before giving a mathematical definition of it, we have to analyze all the requirements that can make a DE unique, also by means of simple examples. It is well known that, locally, any rth order differential equation $\mathcal{E}$ with n independent variables and m dependent ones can be interpreted as a submanifold of the r-jet $J^r(n, m)$ of the trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Let us denote by $\mathrm{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of $\mathcal{E}$. Thus, when saying that $\mathcal{E}$ is uniquely determined by $\mathrm{sym}(\mathcal{E})$, one should fix, as data of the problem, the number of independent and dependent variables, the order of the DE and its dimension as submanifold. For instance, (see also Section 4.1), the unique 5th order ODE admitting the algebra of projective vector fields of $\mathbb{R}^2$ is equation of item 5 of Theorem 4, but also $u_{xx} = 0$ is the unique 2nd order ODE admitting the algebra of projective vector fields of $\mathbb{R}^2$ as Lie algebra of point symmetries. Remaining in the realm of projective algebra, the system $\{y_{xx} = 0, u_{xx} = 0\}$ is uniquely determined by the 15-dimensional projective Lie algebra of $\mathbb{R}^3$, but, as we already said, also the Monge–Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 0$ admits the same 15-dimensional Lie algebra of vector fields as Lie algebra of point symmetries. Both the system $\{y_{xx} = 0, u_{xx} = 0\}$ and $u_{xx}u_{yy} - u_{xy}^2 = 0$ are, in their own class, the only DEs admitting the projective algebra of $\mathbb{R}^3$ as Lie algebra of point symmetries. As the last consideration, we observe that if an equation $\mathcal{E}$ admits a Lie algebra of point symmetries, also an open submanifold of $\mathcal{E}$ admits the same Lie algebra of symmetries, so that when speaking about Lie remarkable equations one should think of them up to inclusion. Bringing all the above observations together, below we formulate a more precise definition of Lie remarkable equations.

Notations and conventions: Throughout the paper, we will use the Einstein summation convention, unless otherwise specified. We will always use the word "symmetry" for "infinitesimal point symmetry". When we speak about a Lie algebra we always mean a *Lie algebra of vector fields* of finite dimension, unless otherwise specified. Finally, if $\mathfrak{s}$ and $\mathfrak{g}$ are Lie algebras, $\mathfrak{s} \leq \mathfrak{g}$ means that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$.

Definition 1. An l-dimensional rth order equation $\mathcal{E} \subset J^r(n, m)$ is called *Lie remarkable* if it is the only l-dimensional rth order equation in $J^r(n, m)$, up to inclusion and up to point transformations, admitting $\mathrm{sym}(\mathcal{E})$ as a Lie symmetry subalgebra.

Below we will shed light on the above definition by means of a simple example. Equation

$$\mathcal{E}_1 : \ u_{xx} = \frac{1}{2}u_x + e^{-2x}u_x^3$$

is not Lie-remarkable. In fact $\mathrm{sym}(\mathcal{E}_1)$ is linearly generated by

$$\partial_u, \qquad \partial_x + u\partial_u, \qquad u\partial_x + \frac{u^2}{2}\partial_u \tag{2}$$

but also the equation

$$\mathcal{E}_2 : \ u_{xx} = \frac{1}{2}u_x$$

admits $\mathrm{sym}(\mathcal{E}_1)$ as a Lie subalgebra of its Lie symmetry algebra. Indeed, $\mathrm{sym}(\mathcal{E}_1) \lneq \mathrm{sym}(\mathcal{E}_2)$, as $\mathrm{sym}(\mathcal{E}_2)$ is isomorphic to the projective Lie algebra of $\mathbb{R}^2$. Thus, $\mathcal{E}_1$ and $\mathcal{E}_2$ are not equivalent. To conclude, $\mathcal{E}_1$ is not Lie-remarkable, whereas $\mathcal{E}_2$ is.

Of course, an abstract Lie algebra can be realized, in terms of vector fields, in different non-equivalent ways. For instance, S. Lie [29] investigated the possible realizations of the non-commutative Lie algebra of dimension 2 (i.e., the Lie algebra spanned by two elements X and Y such that $[X, Y] = X$) as Lie algebra of vector fields on $\mathbb{R}^2$. He showed that, almost every