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Projective-geometric aspects of homogeneous third-order Hamiltonian operators

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Dedicated to the memory of Professor Yavuz Nutku (1943–2010)

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ABSTRACT

We investigate homogeneous third-order Hamiltonian operators of differential-geometric type. Based on the correspondence with quadratic line complexes, a complete list of such operators with $n \leq 3$ components is obtained.

1. Introduction

First-order homogeneous Hamiltonian operators were introduced in [1] in the study of one-dimensional systems of hydrodynamic type.[1] It was demonstrated that these operators are parametrized by flat pseudo-Riemannian metrics. Higher-order operators were subsequently defined in [2]. The structure of homogeneous second-order Hamiltonian operators was investigated in [3,4], see also [5,6].

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[1] They are also known as differential-geometric, or Dubrovin–Novikov brackets.

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In this paper we address the problem of classification of homogeneous third-order Hamiltonian operators of differential-geometric type [7,4,8,5,9],

$$P = g^{ij}D^3 + b^{ij}_k u^k_x D^2 + (c^{ij}_k u^k_{xx} + c^{ij}_{km} u^k_x u^m_x)D + d^{ij}_k u^k_{xxx} + d^{ij}_{km} u^k_{xx} u^m_x + d^{ij}_{kmn} u^k_x u^m_x u^n_x. \tag{1}$$

Here u^i, $i = 1, \ldots, n$, are the dependent (field) variables, and the coefficients $g^{ij}, \ldots, d^{ij}_{kmn}$ depend on u^i only; D stands for the total derivative with respect to x. Homogeneity is understood as follows: the independent variable x has order -1, the dependent variables u^i have order 0, so that the order of u^i_x and D is 1, etc. The operator P is Hamiltonian if and only if it is formally skew-adjoint, $P^* = -P$, and its Schouten bracket vanishes, $[P, P] = 0$. Equivalently, the corresponding Poisson bracket,

$$\{F_1, F_2\} = \int \frac{\delta F_1}{\delta u^i} P^{ij} \frac{\delta F_2}{\delta u^j} dx,$$

must be skew-symmetric, and satisfy the Jacobi identity. We restrict our considerations to the non-degenerate case, $\det g^{ij} \neq 0$. Operators (1) are form-invariant under point transformations of the dependent variables, $u = u(\tilde{u})$. Under point transformations, the coefficients of (1) transform as differential-geometric objects. For instance, g^{ij} transforms as a $(2, 0)$-tensor, so that its inverse g_{ij} defines a pseudo-Riemannian metric (that is not flat in general), the expressions $-\frac{1}{3}g_{js}b^{si}_k$, $-\frac{1}{3}g_{js}c^{si}_k$, $-g_{js}d^{si}_k$ transform as Christoffel symbols of affine connections, etc. It was conjectured in [10] that the last connection, $\Gamma^i_{jk} = -g_{js}d^{si}_k$, must be symmetric and flat; this was confirmed in [7], see also [4]. Therefore, there exists a coordinate system (flat coordinates) such that Γ^i_{jk} vanish. These coordinates are determined up to affine transformations. We will keep for them the same notation u^i, note that u^i are nothing but the densities of Casimirs of the corresponding Hamiltonian operator (1). In the flat coordinates the last three terms in (1) vanish, leading to the simplified expression [5],

$$P = D\left(g^{ij}D + c^{ij}_k u^k_x\right)D. \tag{2}$$

This operator is Hamiltonian if and only if the coefficients g^{ij} and c^{ij}_k satisfy the following relations:

$$g^{ij}_{,k} = c^{ij}_k + c^{ji}_k, \tag{3a}$$

$$c^{ij}_s g^{sk} = -c^{kj}_s g^{si}, \tag{3b}$$

$$c^{ij}_s g^{sk} + c^{jk}_s g^{si} + c^{ki}_s g^{sj} = 0, \tag{3c}$$

$$c^{ij}_{s,m} g^{sk} = c^{ik}_s c^{sj}_m - c^{ki}_s c^{sj}_m - c^{kj}_s g^{si}_{,m}. \tag{3d}$$

Here (3a) is equivalent to $P^* = -P$, while (3b)–(3d) are equivalent to $[P, P] = 0$. These conditions are invariant under affine transformations of the flat coordinates. It is useful to rewrite the above system in low indices. Introducing $c_{ijk} = g_{iq}g_{jp}c^{pq}_k$ one obtains [8]:

$$g_{mn,k} = -c_{mnk} - c_{nmk}, \tag{4a}$$

$$c_{mnk} = -c_{mkn}, \tag{4b}$$

$$c_{mnk} + c_{nkm} + c_{kmn} = 0, \tag{4c}$$

$$c_{mnk,l} = -g^{pq}c_{pml}c_{qnk}. \tag{4d}$$

Our main observation is that the metric g satisfying Eqs. (4) must be the Monge metric of a quadratic line complex. Since complexes of lines belong to projective geometry, Eqs. (4) should be invariant under the full projective (rather than affine) group. We demonstrate that this is indeed the case. Based on the projective classification of quadratic line complexes in $\mathbb{P}^3$ into eleven Segre types [11], we give a complete list of three-component Hamiltonian operators.

The structure of the paper is as follows. After discussing known examples of third-order homogeneous Hamiltonian operators in Section 2, we summarize our main results in Section 3. In Section 4 we establish a link between homogeneous third-order Hamiltonian operators and Monge metrics/quadratic line complexes. This indicates that the theory is essentially projectively-invariant (Section 5), and leads to the classification results presented in Section 6.

All computations were performed with the software package CDIFF [12] of the REDUCE computer algebra system [13].

2. Examples

To the best of our knowledge, all interesting examples of integrable systems possessing Hamiltonian structures of the form (1) come from the theory of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of 2D topological field theory. These are integrable PDEs of Monge–Ampère type that acquire a Hamiltonian formulation upon transformation into hydrodynamic form [14].