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# The four-dimensional Martínez Alonso–Shabat equation: Reductions and nonlocal symmetries

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ABSTRACT

We consider the four-dimensional integrable Martínez Alonso–Shabat equation and list three integrable three-dimensional reductions thereof. We also present a four-dimensional integrable modified Martínez Alonso–Shabat equation together with its Lax pair.

We further construct an infinite hierarchy of commuting nonlocal symmetries (and not just the shadows, as it is usually the case in the literature) for the Martínez Alonso–Shabat equation.



## 1. Introduction

Consider the four-dimensional Martínez Alonso–Shabat equation

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{1}$$

introduced in [1]. It has [2] a covering defined by system

$$q_y = \lambda\, u_y\, q_x, \qquad q_z = \lambda\, (u_z\, q_x - q_t) \tag{2}$$

with a non-removable parameter $\lambda \neq 0$, and a recursion operator, and is therefore integrable.

Below we present three reductions of (1) to integrable three-dimensional equations: the so-called rdDym equation [3–6], the universal hierarchy equation [1], and Eq. (4) related [7,8] to the ABC equation; see [9,10] for the latter.

Note that eliminating $u$ from the Lax pair (2) for (1) yields an integrable four-dimensional PDE (10) to which we refer as to the *modified* Martínez Alonso–Shabat equation and which can be seen as a four-dimensional generalization of the ABC equation (6). Integrability of (10) is established by presenting a Lax pair (11) for the latter. It would be interesting to study (10) in more detail, e.g. to find a recursion operator for this equation.

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The main goal of the rest of the present paper is to find nonlocal symmetries for Eq. (1) in the covering (12) derived from the Lax pair (2).

Following [11–13] and references therein, recall that a *higher* (or *generalized* [13]) *symmetry* for a partial differential system $\mathcal{E}$ can be identified with its characteristics, which is, roughly speaking, a vector function on an appropriate jet space $J^\infty(\mathcal{E})$ associated with $\mathcal{E}$ that depends on finitely many arguments and satisfies the linearized version of $\mathcal{E}$.

Next, a (differential) *covering* for a partial differential system $\mathcal{E}$ is (see e.g. [14,12,15,16]) an over-determined system $\tilde{\mathcal{E}}$ involving additional dependent variables (which are called *pseudopotentials*) such that $\mathcal{E}$ implies the compatibility conditions of $\tilde{\mathcal{E}}$. A vector function on $J^\infty(\tilde{\mathcal{E}})$ that depends on finitely many arguments and satisfies the linearized version of $\mathcal{E}$ is called [14,17,15,16] a *shadow* (more precisely, a $\tilde{\mathcal{E}}$-shadow) for $\mathcal{E}$.

Thus, in contrast with the symmetry of $\mathcal{E}$, a shadow is allowed to depend on the pseudopotentials and their derivatives. On the other hand, symmetries of $\tilde{\mathcal{E}}$ are called [14,17,15,12,16] *nonlocal symmetries* of $\mathcal{E}$ associated with the covering $\tilde{\mathcal{E}}$.

Infinite-dimensional Lie algebras of nonlocal symmetries are well known to play an important role in the theory of integrable systems and provide a useful tool for the study of the latter; see e.g. [3,16] and references therein. In this connection note (see e.g. [15]) that not every $\tilde{\mathcal{E}}$-shadow can be lifted to a full-fledged nonlocal symmetry for $\mathcal{E}$ associated with $\tilde{\mathcal{E}}$. This fact has profound consequences. In particular, while for the full nonlocal symmetries we can readily define their Lie bracket, this is not quite the case for the shadows.

The results of Sections 4 and 5 of the present paper can now be summarized as follows.

First, using (2) as a starting point we construct a new covering (12) for (1). Expanding the pseudopotential in (12) into the formal Taylor series w.r.t. the spectral parameter $\lambda$ gives a new covering (19) with an infinite number of new pseudopotentials $w_i$ which are the coefficients at the powers of $\lambda$.

We then proceed to construct an infinite hierarchy of commuting nonlocal symmetries for Eq. (1) in this new covering using a technique from [18]. Let us stress that this construction makes heavy use of the fact that the covering (19) can be promoted to a covering (19) + (21b) over the system that consists of (1) and (21a). It should also be mentioned that finding explicit formulas for full-fledged nonlocal symmetries of equations in more than two independent variables rather than mere shadows is quite uncommon; cf. e.g. [19,18] and the discussion at the end of Section 5.

## 2. Reductions of the Martínez Alonso–Shabat equation

It is a remarkable fact that three known integrable three-dimensional PDEs can be obtained as reductions of (1).

### 2.1. The rdDym equation

The reduction $z = x$ yields the rdDym equation [3–6] that arises as the $r \to \infty$ limit of the so-called $r$th dispersionless Harry Dym equation [3]:

$$u_{ty} = u_x\, u_{xy} - u_y\, u_{xx}. \tag{3}$$

The Lax pair (2) after the reduction boils down to the known Lax pair for (3),

$$q_t = (u_x - \lambda^{-1})\, q_x, \qquad q_y = \lambda\, u_y\, q_x.$$

### 2.2. The universal hierarchy equation

Putting $t = y$ in (1) and (2) yields the universal hierarchy equation [1]

$$u_{yy} = u_z\, u_{xy} - u_y\, u_{xz}$$

and its Lax pair

$$q_y = \lambda\, u_y\, q_x, \qquad q_z = \lambda\,(u_z - \lambda\, u_y)\, q_x.$$

### 2.3. An equation related to the ABC equation

Another interesting reduction admitted by (1) arises when we put $z = t$. This produces the equation

$$u_{ty} = u_t\, u_{xy} - u_y\, u_{tx}, \tag{4}$$

which, along with the associated Lax representation,

$$q_t = \lambda\, u_t\, q_x/(\lambda + 1), \qquad q_y = \lambda\, u_y\, q_x, \tag{5}$$

obtained by performing the reduction in question in (2), has already appeared in the literature; see [7,8].

Upon eliminating $u$ from (5) we arrive at the equation

$$q_y\, q_{tx} = (\lambda + 1)\, q_t\, q_{xy} - \lambda\, q_x\, q_{ty}, \tag{6}$$