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The four-dimensional Martínez Alonso–Shabat equation: Differential coverings and recursion operators

Oleg I. Morozov *

Institute of Mathematics and Statistics, University of Tromsø, Tromsø 90-37, Norway

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ABSTRACT

We apply Cartan's method of equivalence to find a contact integrable extension for the structure equations of the symmetry pseudo-group of the four-dimensional Martínez Alonso–Shabat equation. From this extension we derive two differential coverings including coverings with one and two non-removable parameters. Then we apply the same approach to the tangent covering and construct a recursion operator for symmetries of the equation under study.

1. Introduction

We consider the partial differential equation (PDE)

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{1}$$

introduced by L. Martínez Alonso and A.B. Shabat in [1]. This equation has a number of important properties, see, e.g., [2,3]. In this paper we apply the technique of [4] to find a contact integrable extension (CIE) for the structure equations of the symmetry pseudo-group of (1). This CIE produces two differential coverings, [5], for Eq. (1). In the general case the first covering contains two non-removable parameters, but there is a special case when the covering has one such parameter.

Then we employ the approach of [6] to find a Bäcklund auto-transformation for the tangent covering, [7,8], of (1). This transformation provides a recursion operator for symmetries of (1).

2. Coverings of the 4D Martínez Alonso–Shabat equation

Using procedures of Cartan's method of equivalence, [9–14], we compute Maurer–Cartan forms (MCFs) and structure equations for the symmetry pseudo-group of Eq. (1). The structure equations are presented in the Appendix. They contain

* Present address: Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30059 Kraków, Poland. Email: morozov@agh.edu.pl.

E-mail addresses: oimorozov@gmail.com, Oleg.Morozov@uit.no.

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the following MCFS:

$$\theta_3 = u_y^{-1}\,\left(du_y - (u_z\,u_{xy} - u_y\,u_{xz})\,dt - u_{xy}\,dx - u_{yy}\,dy - u_{yz}\,dz\right),$$
$$\theta_4 = \frac{u_{xy}\,u_{yy} - u_y\,u_{xyy}}{u_y\,R}\,\left(du_z - u_{tz}\,dt - u_{xz}\,dx - u_{yz}\,dy - u_{zz}\,dz\right) + \frac{u_y\,u_{xyz} - u_{xy}\,u_{yz}}{R}\,\theta_3,$$
$$\xi^1 = \frac{R\,dt}{a\,u_y^2}, \qquad \xi^3 = a\,u_y\left(dy + \frac{u_y\,u_{xyz} - u_{xy}\,u_{yz}}{u_y\,u_{xyy} - u_{xy}\,u_{yy}}\,dz\right), \qquad \xi^4 = \frac{a\,u_y\,R\,dz}{u_{xy}\,u_{yy} - u_y\,u_{xyy}},$$
$$\xi^2 = \frac{1}{a\,u_y^3}\,\left(\left(u_y\,(u_y\,u_{xyz} - u_{xy}\,u_{yz}) - u_z\,(u_y\,u_{xyy} - u_{xy}\,u_{yy})\right)\,dt + (u_{xy}\,u_{yy} - u_y\,u_{xyy})\,dx\right),$$
$$\eta_4 = \frac{u_y\,du_{xyy} - u_{xy}\,du_{yy} - u_{yy}\,du_{xy}}{u_y\,u_{xyy} - u_{xy}\,u_{yy}} - 2\,\frac{da}{a} + \eta_1 - \frac{(2\,u_y\,u_{xyy} - 3\,u_{xy}\,u_{yy})\,du_y}{u_y\,(u_y\,u_{xyy} - u_{xy}\,u_{yy})},$$
$$\eta_1 = \frac{da}{a} + \frac{1}{u_y}\,\left(u_{xy}\,dx + (u_z\,u_{xy} - u_y\,u_{xz})\,dt\right), \tag{2}$$

where $a \neq 0$ is a parameter and

$$\begin{aligned} R = {} & (u_y\,u_{yz}\,u_{xy}^2\,(3\,u_y\,u_{yz} - 2\,u_z\,u_{yy})\,u_{xyz} - u_y^2\,u_{xy}\,(3\,u_y\,u_{yz} - u_z\,u_{yy})\,u_{xyz}^2 \\ & + u_y^2\,(u_y\,u_{tzz} + u_{zz}\,(u_y\,u_{xz} - u_z\,u_{xy})\,u_{xyy}^2 + u_{xy}^3\,u_z\,u_{yy}\,(u_{yz}^2 - u_{yy}\,u_{zz})) \\ & + u_y\,u_z\,u_{xy}^2\,(u_{yz}^2 - 2\,u_{yy}\,u_{zz} - 2\,u_y^2\,u_{xy}\,(u_z\,u_{yz}\,u_{xyz} - u_{xz}\,u_{yy}\,u_{zz}))\,u_{xyy} \\ & + u_y^3\,u_z\,u_{xyz}^2\,u_{xyy} + u_y\,u_{xy}^2\,(u_{yy}^2\,u_{tzz} - u_{xy}\,u_{yz}^3 + u_{xz}\,u_{yy}^2\,u_{zz}) + u_y^4\,u_{xyz}^3)^{1/2} \\ & \cdot (u_z\,(u_y\,u_{xyy} - u_{xy}\,u_{yy}) - 3\,u_y\,(u_y\,u_{xyz} - u_{xy}\,u_{yz}))^{-1/2}. \end{aligned}$$

We need no explicit expressions for the other MCFS in what follows.

For the structure equations from the Appendix we find CIES, [4], that have the form

$$d\omega_0 = \left(\sum_{i=0}^{4} A_i\,\theta_i + \sum{}^{*} B_{ij}\,\theta_{ij} + \sum_{s=1}^{16} C_s\,\eta_s + \sum_{j=1}^{4} D_j\,\xi^j + \sum_{k=1}^{2} E_k\,\omega_k\right) \wedge \omega_0 + \sum_{j=1}^{4}\left(\sum_{i=0}^{4} F_{ji}\,\theta_i + \sum_{k=1}^{2} G_{jk}\,\omega_k\right) \wedge \xi^j, \tag{3}$$

where $\sum^*$ denotes summation over all $i, j \in \mathbb{N}$ such that $1 \leq i \leq j \leq 4$ and $(i, j) \neq (1, 3)$. The coefficients of (3) are supposed to be either constants or functions of one additional function X_0 with the differential of the form

$$dX_0 = \sum_{i=0}^{4} H_i\,\theta_i + \sum{}^{*} I_{ij}\,\theta_{ij} + \sum_{s=1}^{16} J_s\,\eta_s + \sum_{j=1}^{4} K_j\,\xi^j + \sum_{q=0}^{2} L_q\,\omega_q. \tag{4}$$

We require compatibility of the structure equations from the Appendix and Eqs. (3). This yields an over-determined system of algebraic or ordinary differential equations for the coefficients of (3) and (4). The analysis of these systems gives the following result (the computations were held in the JETS software, [15,16]).

Theorem 1. *The structure equations from the Appendix have no* CIE *(3) with constant coefficients. Each their* CIE *(3), (4) with one additional function X_0 is contact-equivalent to the system*

$$d\omega_0 = (\eta_1 - \omega_1) \wedge \omega_0 + X_0\,(\omega_2 + \theta_4) \wedge \xi^1 + X_0\,\omega_1 \wedge \xi^2 - (\omega_1 + \theta_3) \wedge \xi^3 + \omega_2 \wedge \xi^4, \tag{5}$$
$$dX_0 = X_0\,(\eta_1 - \eta_4) + Y_1\,(\omega_0 - X_0\,\xi^2 + \xi^3) + Y_2\,(X_0\,\xi^1 + \xi^4), \tag{6}$$

where Y_1, Y_2 are arbitrary parameters.

The inverse third fundamental Lie theorem in Cartan's form, [9, Sections 16–24], [10], [17, Sections 16, 19, 20, 25, 26], [18, Sections 14.1–14.3], ensures the existence of the forms ω_0, ω_1, ω_2 and the function X_0 satisfying system (5), (6). Since the MCFS (2) are known explicitly, it is not hard to find the form ω_0. The integration of (5), (6) depends on whether the conditions $Y_1 \equiv 0$ or $Y_1 \not\equiv 0$ hold. In the first case we have

$$\omega_0 = a\,H^{-1}\,q_x^{-1}\,\left(dq - (u_z\,q_x - H^{-1}\,q_z)\,dt - q_x\,dx - H\,u_y\,q_x\,dy - q_z\,dz\right), \tag{7}$$

where $H = H(t, z)$ is a solution to equation

$$H_z + H\,H_t = 0, \tag{8}$$

while in the case of $Y_1 \not\equiv 0$

$$\omega_0 = a\,q^{-1}\,q_x^{-1}\,\left(dq - (u_z\,q_x - q^{-1}\,q_z)\,dt - q_x\,dx - q\,u_y\,q_x\,dy - q_z\,dz\right).$$