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On symplectization of 1-jet space and differential invariants of point pseudogroup

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ABSTRACT

In this paper we suggest an approach to the study of the action of point pseudogroup in (contact) 1-jet space $J^1\mathbb{R}^n$. This approach is based on the following idea. We replace the canonic projection $\pi_{1,0}: J^1\mathbb{R}^n \to J^0(\mathbb{R}^n)$ from 1-jet space to 0-jet space with projective action on the fiber by some bundle, which is called *symplectization*, with the linear action on the fiber. This makes it possible to apply methods of studying the linear actions of algebraic groups and to find the set of independent differential invariants. Finally, we rewrite the action of point pseudogroup in terms of these invariants and classify the orbits of point pseudogroup action. We also give two examples of application of this method. In the first example we obtain classification of contact vector fields in $J^1\mathbb{R}^n$, and in the second we obtain classification of symbols of linear differential operators.

1. Introduction

Studying of 1-jet space $J^1\mathbb{R}^n$ of smooth functions $f: \mathbb{R}^n \to \mathbb{R}$ and different structures on it is of great interest because of many applications to the theory of differential equations (ODE's and PDE's). The main object is *contact structure* on 1-jet space (which is defined by Cartan distribution), because its maximal integral manifolds are solutions of differential equations.

It is also natural to study the actions of different pseudogroups, which preserve contact structure. Usually, two pseudogroups are considered (see [1]):

- *pseudogroup of contact transformations*, i.e. pseudogroup of diffeomorphisms of 1-jet space $J^1\mathbb{R}^n$, which preserve contact structure;
- *pseudogroup of point transformations*, i.e. pseudogroup of contact diffeomorphisms of 1-jet space $J^1\mathbb{R}^n$, which preserve base $J^0\mathbb{R}^n$.

Classic result of Sophus Lie claims that all first order differential equations in the neighborhood of non-singular point are point-equivalent. So, it is natural to study the actions of contact and point pseudogroups on the other objects connected with 1-jet space $J^1\mathbb{R}^n$.

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It is easy to show that all regular smooth functions on 1-jet space are contact-equivalent. In [2] point classification of regular smooth functions on 1-jet space $J^1\mathbb{R}^n$ was obtained (note that each smooth function on $J^1\mathbb{R}^n$ can be considered as first order non-linear differential operator). In [3] this result was generalized on the cases of point and contact classification of smooth functions on the k-jet spaces for all k.

The question of classification of contact vector fields is more difficult. It can be shown that all contact vector fields are contact equivalent in a neighborhood of non-singular point.

It was proved by V. Lychagin in 1977 (see [4]) that under some conditions the orbit of contact vector field in a neighborhood of singular point with respect to the action of contact pseudogroup is defined by finite jet of this field.

In this paper we suggest a new method for studying the action of point pseudogroup on the objects in 1-jet space $J^1\mathbb{R}^n$. Namely, we replace the canonic projection $\pi_{1,0}\colon J^1(\mathbb{R}^n) \to J^0(\mathbb{R}^n)$ by some bundle (which is called *symplectization of 1-jet space* $J^1\mathbb{R}^n$) with the linear action of point pseudogroup on its fibers. Then we apply the results of [5] and get the set of differential invariants for this action. Finally, we use the idea from work [6] and rewrite the action of point pseudogroup in terms of these differential invariants.

We give two examples to illustrate these ideas. Namely, we study the action of point pseudogroup on the following objects:

- contact vector field;
- symbols of linear differential operators.

We obtain a set of differential invariants of these objects and their point classification.

2. Necessary definitions and notations

All our constructions are geometrical, i.e. they do not depend on coordinates. Nevertheless, we will provide them in both geometric and coordinate terms.

2.1. Jet space

Let $\mathbb{R}^n$ be a real n-dimensional space. We recall that 1-*jet* $[f]^1_a$ *of function* $f \in C^\infty(\mathbb{R}^n)$ *at point* $a \in \mathbb{R}^n$ is the equivalent class of the graphs $L_{\widetilde{f}}$ of functions $\widetilde{f}$ in 0-jet space $J^0\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}$, which are tangent to the graph L_f in point $(a, f(a))$. It means that 1-jet $[f]^1_a$ is defined by point a, by the values of function f and all partial derivatives of f in this point. Hence, in coordinates 1-jet $[f]^1_a$ can be written in the following way:

$$[f]^1_a = \Big(a, f(a), f_{x_1}(a), \dots, f_{x_n}(a)\Big),$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$.

The set of all 1-jets of all smooth functions in all points is called 1-*jet space* and is denoted as $J^1\mathbb{R}^n$. Canonical coordinates on this space will be denoted as

$$(\mathbf{x}, y, \mathbf{y}'), \quad \text{where } \mathbf{x} := (x_1, \dots, x_n) \text{ are the coordinates on } \mathbb{R}^n \text{ and } \mathbf{y}' := (y_1, \dots, y_n).$$

By definition, one has $y([f]^1_a) = f(a)$ and $y_i([f]^1_a) = f_{x_i}(a)$.

2.2. Cartan distribution

It is well known, that 1-jet space $J^1\mathbb{R}$ has the natural contact structure, which is called *Cartan distribution*.

For each function $f \in C^\infty(\mathbb{R}^n)$ one can define its 1-*graph* $L^1_f = \{[f]^1_a : a \in \mathbb{R}^n\}$. For a given 1-jet θ, let us consider planes $\mathcal{C}_\theta \subset T_\theta(J^1\mathbb{R}^n)$ generated by tangent spaces $T_\theta L^1_f$ for all 1-graphs L^1_f, which pass through 1-jet θ. The distribution $\mathcal{C}\colon \theta \mapsto \mathcal{C}_\theta$ is called *Cartan distribution*.

In coordinates 1-graph of function f can be written as

$$L^1_f := \{(a, f(a), f_{x_1}(a), \dots, f_{x_n}(a)) : a \in \mathbb{R}^n\}.$$

Hence, Cartan distribution can be generated by differential 1-form $\varkappa := dy - \mathbf{y}'\,d\mathbf{x}$, which is called *Cartan form*: $\mathcal{C} = \ker \varkappa$.

Remark. Here and further by multiplication of vectors we assume the sum of products of corresponding coordinates:

$$\varkappa = dy - \mathbf{y}'\,d\mathbf{x} = dy - y_1 dx_1 - \cdots - y_n dx_n.$$

Remark. Differential Cartan 1-form $\varkappa$ is defined up to the multiplication on non-zero smooth function $\mu \in C^\infty(J^1\mathbb{R}^n)$.

It is clear that

$$\dim \mathcal{C} = 2n = \dim J^1\mathbb{R}^n - 1 \quad \text{and} \quad \varkappa \wedge (d\varkappa)^n = dy \wedge (d\mathbf{x} \wedge d\mathbf{y}')^{\wedge n} \neq 0,$$

i.e. Cartan distribution defines a contact structure on 1-jet space $J^1\mathbb{R}^n$.