Journal of Geometry and Physics 85 (2014) 88–98

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On Lipschitz solutions of the constant astigmatism equation

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ARTICLE INFO

Article history:
Received 22 January 2014
Received in revised form 30 April 2014
Accepted 19 May 2014
Available online 28 May 2014

MSC:
35Q53
35Q74
53A05

Keywords:
Constant astigmatism equation
Constant astigmatism surface
Orthogonal equiareal pattern
sine–Gordon equation
Symmetry invariant solution
Slip-line field

ABSTRACT

We show that the solutions of the constant astigmatism equation that correspond to a class of surfaces found by Lipschitz in 1887, exactly match the Lie symmetry invariant solutions and constitute a four-dimensional manifold. The two-dimensional orbit space with respect to the Lie symmetry group is described. Our approach relies on the link between constant astigmatism surfaces and orthogonal equiareal patterns. The counterpart sine–Gordon solutions are shown to be Lie symmetry invariant as well.

1. Introduction

This paper continues and complements our study [1,2] of the constant astigmatism equation

$$z_{yy} + (1/z)_{xx} + 2 = 0. \tag{1}$$

This is the Gauss equation for surfaces immersed in three-dimensional Euclidean space and characterised by a constant difference between the principal radii of curvature (named surfaces of constant astigmatism by analogy with geometrical optics). Despite its nineteenth-century roots, see [3] for a historical account, Eq. (1) appears to have been missed until [1]. The same equation states that the metric

$$z\, dx^2 + (1/z)\, dy^2 \tag{2}$$

is of Gaussian curvature +1 and, therefore, is an admissible metric of the sphere; cf. Bianchi [4, Section 375, Eq. (20)] in the context of pseudospherical congruences.

The two interpretations of Eq. (1) are closely related. Indeed, formula (2) characterises the third fundamental forms of the constant astigmatism surfaces, see [2]. It follows that solutions $z(x, y)$ of Eq. (1) must be positive valued to be geometrically meaningful.

One can only speculate why constant astigmatism surfaces fell into oblivion throughout most of the twentieth century. Our own motivation behind the revival of the subject is to pursue application of contemporary solution methods to Eq. (1);

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http://dx.doi.org/10.1016/j.geomphys.2014.05.020

this interest is shared with the authors of [5,6]. A review of known exact solutions is not only a useful prerequisite, but also a debt to the history.

The most important nineteenth-century result of Bianchi [7,4] establishes a correspondence between constant astigmatism and pseudospherical surfaces, the latter being evolutes of the former. This link leads to transformation formulas between solutions $z(x, y)$ of Eq. (1) and $q(\xi, \eta)$ of the sine–Gordon equation

$$q_{\xi\eta} = \sin q, \tag{3}$$

investigated and exploited elsewhere [1,2].

In 1887 Lipschitz [8] presented a completely different class of surfaces of constant astigmatism in terms of spherical coordinates of the Gaussian image. It is not easy to see what are the corresponding solutions $z(x, y)$ and $q(\xi, \eta)$. However, Lipschitz's derivation points to a hidden relationship to orthogonal equiareal patterns on the sphere. By the latter we mean coordinate systems such that the sphere's metric acquires the form (2). A physical interpretation, see [2], says that the principal stress lines of a plastic round sphere under the Tresca yield condition constitute an orthogonal equiareal pattern, cf. [9,10] for the planar case. Moreover, the slip-lines are curves forming an angle of $\pi/4$ with the principal stress lines. Remarkably enough, slip line fields on the sphere are determined by the sine–Gordon solutions $q(\xi, \eta)$; see [2].

In the present paper, by re-deriving the Lipschitz class in terms of the orthogonal equiareal patterns, we easily compute the corresponding solutions $z(x, y)$ of the constant astigmatism equation. The solutions turn out to be double-valued and constituting a four-dimensional manifold. Moreover, we show that they exactly match the solutions invariant with respect to the Lie symmetries of the constant astigmatism equation (see below). The above-mentioned manifold of solutions is acted upon by the Lie symmetry group, the orbit space being two-dimensional. Finally, in Section 3 we find the corresponding sine–Gordon solutions; they turn out to be symmetry invariant as well, and actually well known.

The Lie symmetries of the constant astigmatism equation (1) are the translations $\mathfrak{T}^x : (x, y, z) \mapsto (x + t, y, z)$, $\mathfrak{T}^y : (x, y, z) \mapsto (x, y + t, z)$, and the scaling $\mathfrak{S} : (x, y, z) \mapsto (e^{-t}x, e^{t}y, e^{2t}z)$, where t is a real parameter. This is an exhaustive list of one-parametric groups of contact transformations, obtained through a routine computation of infinitesimal generators (generating functions)

$$\mathfrak{t}^x = z_x, \qquad \mathfrak{t}^y = z_y, \qquad \mathfrak{s} = xz_x - yz_y + 2z \tag{4}$$

(see [11] for the general theory or [12] for an overview). Also useful will be the discrete symmetry $\mathfrak{I} : x \mapsto y, y \mapsto x, z \mapsto 1/z$, called the *involution*.

For future reference, we single out the solutions

$$z = \alpha^2 - (y + \beta)^2, \qquad z = \frac{1}{\alpha^2 - (x - \beta)^2}, \tag{5}$$

where α, β denote arbitrary real constants. They are easy to obtain as the solutions independent of x or y, respectively. Following [1], we call them the *von Lilienthal solutions*, since they correspond to the constant astigmatism surfaces of revolution studied by von Lilienthal [13].

2. Lipschitz solutions

Consider the Gaussian sphere $\mathbf{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ parameterised by the colatitude (polar angle) $0 \le \theta \le \pi$ and the longitude $0 \le \phi \le 2\pi$. To specify an orthogonal equiareal pattern, we let θ, ϕ denote yet unknown functions of parameters x, y. Lipschitz defines a position angle (Stellungswinkel) to be the angle ω between the meridians $\mathbf{n}_\theta$ and the pattern's lines $y = \text{const}$, i.e., $\mathbf{n}_x = \phi_x \mathbf{n}_\phi + \theta_x \mathbf{n}_\theta$. The Lipschitz class is specified by letting the position angle ω depend solely on the colatitude θ. The corresponding solution of the constant astigmatism equation will be called the *Lipschitz solution*. As we shall prove immediately below, a generic Lipschitz solution is two-valued and depends on four real parameters.

Theorem 1. *The general Lipschitz solution of the constant astigmatism equation (1) depends on four real parameters h_{11}, h_{10}, h_{01}, h_{00} and is a nonzero root of the quadratic polynomial*

$$h_y^2 z^2 + (h^2 - 1)z + h_x^2, \tag{6}$$

where

$$h = h_{11}xy + h_{10}x + h_{01}y + h_{00},$$
$$h_y = h_{11}x + h_{01}, \qquad h_x = h_{11}y + h_{10},$$

under the condition that h is not a constant (i.e., at least one of the coefficients h_{11}, h_{10}, h_{01} is not zero).

Proof. Expressing the metric of the sphere in two ways, we have

$$d\theta^2 + \sin^2\theta \, d\phi^2 = z\,dx^2 + (1/z)\,dy^2,$$