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On differential invariants of actions of semisimple Lie groups

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ABSTRACT

In this paper we suggest an approach to the study of orbits of actions of semisimple Lie groups in their irreducible complex representations. This approach is based on differential invariants on the one hand, and on geometry of reductive homogeneous spaces on the other hand. According to Borel–Weil–Bott theorem, every irreducible representation of semisimple Lie group is isomorphic to the action of this group on the module of holomorphic sections of some one-dimensional bundle over homogeneous space. Using this, we give a complete description of the structure of the field of differential invariants for this action and obtain a criterion, which separates regular orbits.

0. Introduction

This paper concludes the series of papers [1–4], where differential invariants are used for the separation of the orbits of algebraic Lie groups on the spaces of homogeneous forms. In the present paper we solve a general problem of classification of the orbits for the action of arbitrary semisimple complex Lie group in its irreducible representation.

Let G be a connected semisimple complex Lie group, and let

$$\rho_\lambda : G \to \mathrm{GL}(V)$$

be its irreducible representation with highest weight λ (see [5]).

First, let us fix a Borel subgroup B in group G and consider homogeneous complex flag manifold $M := G/B$.

Secondly, consider the action $B : G$ of Borel group B on G by the right shifts:

$$g \mapsto gb^{-1},$$

where $g \in G$ and $b \in B$.

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Finally, let us define the bundle product $E := G \times_B \mathbb{C} = G \times \mathbb{C}/\sim$, where the equivalence relation $\sim$ is defined by the following:

$$(g, c) \sim (gb^{-1}, \chi_\lambda(b)c),$$

and where $\chi_\lambda \in \mathfrak{X}(T)$ is the character corresponding to the highest weight λ of the maximal torus $T \subset B$.

We introduce one-dimensional bundle

$$\pi^\lambda : E \to M, \qquad \pi^\lambda(g, c) = gB.$$

Holomorphic sections of this bundle are just holomorphic functions $f : G \to \mathbb{C}$, which satisfy the relation

$$f(gb) = \chi_\lambda(b)f(g),$$

for all $g \in G$ and $b \in B$.

Group G acts in bundle π^λ by left shifts. This action prolongs to the action on the space of holomorphic sections of bundle π^λ:

$$g(f)(g') = f(g^{-1}g').$$

According to Borel–Weil–Bott theorem (see, for example, [6]), if λ is dominant weight of group G, then this action is isomorphic to representation ρ_λ.

Therefore, the study of orbits of irreducible representations of semisimple complex Lie groups with the highest weight λ is equivalent to the study of the orbits of these actions on the space of holomorphic sections of bundle π^λ.

Let us illustrate this idea in case $G = \mathrm{SL}_2(\mathbb{C})$ (see also [1]). It is known (see, for example, [5,7]), that dominant weights of group $\mathrm{SL}_2(\mathbb{C})$ equal $\lambda = \frac{n}{2}\alpha$, where α is the positive root of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and $n \geqslant 0$ is a non-negative integer. The Borel group $B = \mathrm{B}_2(\mathbb{C})$ consists of upper-triangular matrices, and character χ_λ acts on it in the following way:

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^n.$$

Then

$$M = \mathrm{SL}_2/\mathrm{B}_2 \simeq \mathbb{C}P^1.$$

If we denote the homogeneous coordinates on M by $(x : y)$, then holomorphic sections of bundle π^λ are just homogeneous polynomials of degree n in variables x and y.

Thus, the study of invariants of representations of group $\mathrm{SL}_2(\mathbb{C})$ is reduced to the classification $\mathrm{SL}_2(\mathbb{C})$-orbits of binary forms. This case was considered in [1], where binary forms were considered as solutions of the Euler differential equation.

In this paper we suggest another approach to study of invariants of irreducible representations for semisimple Lie groups based on the Borel–Weil–Bott theorem. Namely, we consider jet space of the section of bundle π^λ, then we describe the differential invariant field of the G-action on the jets of sections and, finally, obtain the criterion, which separates G-orbits of the regular sections of bundle π^λ.

1. Definitions and notations

In this section we introduce basic notations and recall necessary definitions.

1.1. Compact real form

To study invariants of the group G on the module of holomorphic sections of bundle π^λ, we use the following trick.

Let K be the compact real form of group G (see [8]), $\mathfrak{k}$ be its Lie algebra and $T := K \cap B$ be its maximal torus. Then $M \simeq K/T$, $E \simeq K \times_T \mathbb{C}$, and holomorphic sections of $\pi^\lambda : E \to M$ can be considered as functions $f : K \to \mathbb{C}$ such that

$$f(kt) = \chi_\lambda(t)f(k) \quad \text{for all } k \in K \text{ and } t \in T.$$

It follows from the unitary trick (see, for example, [8]) that rational differential G-invariants coincide with rational differential K-invariants. Hence, we shall study the invariants of K-action on the module of holomorphic sections.

1.2. Decomposition of Lie algebras $\mathfrak{k}$ and $\mathfrak{m}$

Also note, that torus $\mathfrak{t}$ defines the decomposition of algebra $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{k}_\alpha = \mathfrak{k}_- \oplus \mathfrak{t} \oplus \mathfrak{k}_+, \qquad \mathfrak{m} = \bigoplus_{\alpha \in \Phi_+} \Pi_\alpha,$$

where

$$\mathfrak{k}_\pm := \bigoplus_{\alpha \in \Phi_\pm} \mathfrak{k}_\alpha, \qquad \Pi_\alpha := \mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha},$$

and $\mathfrak{m}$ is the tangent space to manifold M at T, Φ, $\Phi_\pm$ are the root system and the sets of positive/negative roots.