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Symmetries of filtered structures via filtered Lie equations

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ABSTRACT

We bound the symmetry algebra of a vector distribution, possibly equipped with an additional structure, by the corresponding Tanaka algebra. The main tool is the theory of weighted jets.

1. Introduction and the main result

Consider a manifold M and a non-holonomic (bracket generating) vector distribution $\Delta \subset TM$, possibly equipped with an additional structure, like sub-Riemannian metric or conformal or CR-structure. A specification of these filtered structures is provided below.

Given such a structure, a sheaf of graded Lie algebras $\mathfrak{g} = \oplus \mathfrak{g}_i$ is naturally associated with it. If we consider only the distribution Δ, then $\mathfrak{m}(x) = \mathfrak{g}_-(x)$ is the well-known graded nilpotent Lie algebra (GNLA: nilpotent approximation or Carnot algebra) at $x \in M$, and $\mathfrak{g}(x)$ is its Tanaka prolongation (Tanaka algebra). If an additional structure on Δ is given, then $\mathfrak{g}_0$ or some higher $\mathfrak{g}_i$ $(i > 0)$ is reduced and the algebra is further prolonged. In any case for a filtered structure $\mathcal{F}$ on M we associate its sheaf of Tanaka algebras $\mathfrak{g}(x),\ x \in M$.

Theorem 1. *The symmetry algebra $\mathcal{S}$ (possibly infinite-dimensional) of a filtered structure $\mathcal{F}$ has the natural filtration with the associated grading $\mathfrak{s}$ naturally injected into $\mathfrak{g}(x)$ for any regular point $x \in M$. In particular,*

$$\dim \mathcal{S} \leq \sup_M \dim \mathfrak{g}(x).$$

Provided that the filtered structure is of finite type ($\mathfrak{g}_\kappa(x) = 0$ for some $\kappa > 0$ and all $x \in M$), the right-hand-side can be changed to $\inf_M \dim \mathfrak{g}(x)$.

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If Δ is considered without an extra structure, then this statement was proved in [1] by studying the Lie equation considered as a submanifold in the usual jet-space. In addition to the above mentioned type structures we can impose curvature of the structure as a reduction of $\mathfrak{g}$ (see Remark 3 in Section 3 on the issue of regularity), thus essentially restricting the symmetry algebra, yielding a bound on the gap between the maximal and the next (submaximal) dimensions of the possible symmetry algebras of the given type structures [2]. In the context of parabolic geometries (in the complex-holomorphic or split-real smooth cases) this gap was fully computed using the above idea in [3].

Regularity for the points x in Theorem 1 is defined via the Lie equations in Section 4. When $\sup\dim\mathfrak{g}(x)$ is finite (that is $\mathcal{F}$ is of finite type) or the filtered structure $\mathcal{F}$ is analytic, then the set of regular points is open and dense; in general a generic point is regular.

In this paper we consider weighted jets and relate them to Tanaka algebras. On this way we obtain another proof of Theorem 1 of [1] and get a more general result.

2. Tanaka algebra of a distribution with a structure

Given a distribution Δ its *weak derived flag* $\{\Delta_i\}_{i>0}$, with $\Delta_1 = \Delta$, is given via the module of its sections by $\Gamma(\Delta_{i+1}) = [\Gamma(\Delta), \Gamma(\Delta_i)]$. The distribution Δ will be assumed *completely non-holonomic*, meaning there exists a natural number ν such that $\Delta_\nu = TM$.

The quotient $\mathfrak{g}_i = \Delta_{-i}/\Delta_{-i-1}$ (we let $\Delta_0 = 0$) evaluated at the points $x \in M$ is not a vector bundle in general (rank needs not be constant); however its local sections form a sheaf and the module of its global sections will be denoted by $\Gamma(\mathfrak{g}_i)$ (similarly for other sheafs).

At every point $x \in M$ the vector space $\mathfrak{m} = \oplus_{i<0}\,\mathfrak{g}_i$ has a natural structure of graded nilpotent Lie algebra. The bracket on $\mathfrak{m}$ is induced by the commutator of vector fields on M. Δ is called *strongly regular* if the GNLA $\mathfrak{m} = \mathfrak{m}_x$ does not depend on the point $x \in M$.

The Tanaka prolongation $\mathfrak{g} = \hat{\mathfrak{m}}$ is the graded Lie algebra with negative graded part $\mathfrak{m}$ and non-negative part defined successively by (see [4] for discussion and interpretation of this prolongation)

$$\mathfrak{g}_k = \left\{ u \in \bigoplus_{i<0} \mathfrak{g}_{k+i} \otimes \mathfrak{g}_i^* : u([X,Y]) = [u(X),Y] + [X,u(Y)],\ X,Y \in \mathfrak{m} \right\}.$$

Since Δ is bracket-generating, the algebra $\mathfrak{m}$ is fundamental, i.e. $\mathfrak{g}_{-1}$ generates the whole GNLA $\mathfrak{m}$, and therefore the grade k homomorphism u is uniquely determined by the restriction $u : \mathfrak{g}_{-1} \to \mathfrak{g}_{k-1}$.

At every point $\mathfrak{g} = \oplus\mathfrak{g}_i$ is naturally a graded Lie algebra, called the *Tanaka algebra* of Δ (the bracket is induced by the commutator of vector fields). To indicate dependence on the point $x \in M$, we will write $\mathfrak{g} = \mathfrak{g}(x)$ or $\mathfrak{g}_x$ (also the value of a vector field Y at x will be denoted by Y_x).

In addition to introducing the Lie algebra $\mathfrak{g}$, which majorizes the symmetry algebra of Δ, the paper [5] contains the construction of an important ingredient to the equivalence problem—an absolute parallelism on the prolongation manifold $\mathcal{G}_i$ of the structure, provided it is strongly regular.

Distribution is locally flat if the structure functions of the absolute parallelism vanish. Then the distribution Δ is locally diffeomorphic to the standard model on the Lie group corresponding to $\mathfrak{m}$, see [5].

The prolongation manifolds are the total spaces of the bundles $\mathcal{G}_i \to M$ (in the strongly regular case). For instance, the fiber of $\mathcal{G}_0$ over x consists of all grading preserving isomorphisms of Lie algebras $u_0 : \mathfrak{m} \to \mathfrak{m}_x$, where $\mathfrak{m}$ is an abstract GNLA of the same type as $\mathfrak{m}_x$. Denoting by $\mathrm{Aut}_0(\mathfrak{m})$ the group of grading preserving automorphisms of $\mathfrak{m}$, we conclude that $\mathcal{G}_0$ is a principal $\mathrm{Aut}_0(\mathfrak{m})$-bundle over M; the tangent to the fiber is the Lie algebra $\mathfrak{der}_0(\mathfrak{m})$ of grading preserving derivations of $\mathfrak{m}$. The fiber of $\mathcal{G}_1$ over $u_0 \in \mathcal{G}_0$ consists of the adapted frames $u_1 : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ that uniquely extend to the grading -1 maps $u_1 : \mathfrak{m} \to \mathfrak{m} \oplus \mathfrak{g}_0$ etc., see [6] for details.

The idea of constructing the frame bundle can be pushed to the general non-strongly regular case. Here an abstract reference algebra $\mathfrak{m}$ is lacking[1] and the 0-frames are grading preserving isomorphisms $u_0 : \mathfrak{m}_x \to \mathfrak{m}_x$. Thus $\mathcal{G}_0$ is a fiber bundle (having a distinguished 'identity' section), with the fiber over x being the graded group $\mathrm{Aut}_0(\mathfrak{m}_x)$. Again the fiber of $\mathcal{G}_1$ over $u_0(x) \in \mathcal{G}_0$ is parametrized by the adapted frames $u_1 : \mathfrak{g}_{-1}(x) \to \mathfrak{g}_0(x)$ etc. The fibers of $\mathcal{G}_i \to \mathcal{G}_{i-1}$ for $i > 0$ are isomorphic to $\mathfrak{g}_i$.

Notice that in general, the prolongation manifolds $\mathcal{G}_i$ can be singular, when the fibers have varying dimensions. The structure is called *regular* at the point x if for every i the spaces $\mathfrak{g}_i(y)$ have constant ranks and vary smoothly in y from a neighborhood of x (the size of which can depend on i). The structure is called *regular* if it is regular at every point on M. It is clear that strongly regular distributions have regular prolongations. For regular distributions the above $\mathcal{G}_i$ are smooth fiber bundles over M.

The following statement is similar to [1, Lemma 6] and [3, Lemma 4.2.4], so its proof is omitted.

Proposition 1. *For every i the set of points x, where* $\mathrm{rank}\,\mathfrak{g}_i(x)$ *is locally constant is open and dense in M.* □

[1] This manifests lacking of a canonical absolute parallelism, though it is possible to construct a parallelism respected by any automorphism.