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On stability of equivariant minimal tori in the 3-sphere

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ABSTRACT

We prove that amongst the equivariant constant mean curvature tori in the 3-sphere, the Clifford torus is the only local minimum of the Willmore energy. All other equivariant minimal tori in the 3-sphere are local maxima of the Willmore energy.

0. Introduction

Integrable systems methods have played an important role in exhibiting the complexity and wealth of examples of constant mean curvature (cmc) surfaces in 3-dimensional space forms [1–3]. Associated to a cmc torus is a hyperelliptic Riemann surface called its spectral curve, whose genus g is called the spectral genus. For every genus there are infinitely many examples, and in fact even restricting to minimal tori in the 3-sphere, there are infinitely many examples when $g \geq 1$, see [4]. For equivariant cmc tori the spectral genus is either $g = 0$ or $g = 1$, see [5]. For $g = 0$ the cmc tori are all flat and thus homogeneous, and the Clifford torus is the only minimal example. When $g = 1$, the conformal factor of the metric is an elliptic function of one real variable, and the corresponding cmc tori all have a 1-parameter family of symmetries, (see e.g. Fig. 1.2). Here we consider equivariant minimal tori and their stability under the Willmore energy. We show that while the Clifford torus is stable, all the spectral genus $g = 1$ minimal tori in the 3-sphere $\mathbb{S}^3$ are unstable. In a previous paper [6] we studied the moduli of equivariant cmc tori in $\mathbb{S}^3$ by flowing through cmc tori using the isoperiodic deformation [7]. Using this flow we here obtain the following.

Theorem. *Amongst the equivariant* cmc *tori in the* 3*-sphere, the Clifford torus is the only local minimum of the Willmore energy. All other equivariant minimal tori are local maxima of the Willmore energy.*

1. Homogeneous tori

In order to set notations we first recall how to obtain a cmc surface in the round 3-sphere from a solution of the sinh-Gordon equation. The Maurer–Cartan-equations

$$2\, d\alpha_\lambda + [\, \alpha_\lambda \wedge \alpha_\lambda\,] = 0 \quad \text{for all } \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \tag{1.1}$$

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are an integrability condition. In our setting we define for concreteness

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z\,dz - u_{\bar z}\,d\bar z & \mathbb{i}\,\lambda^{-1}e^u\,dz + \mathbb{i}\,e^{-u}\,d\bar z \\ \mathbb{i}\,e^{-u}\,dz + \mathbb{i}\,\lambda\,e^u\,d\bar z & -u_z\,dz + u_{\bar z}\,d\bar z \end{pmatrix} \tag{1.2}$$

for some smooth function $u : \mathbb{C} \to \mathbb{R}$. The matrix 1-form α_λ takes values in $\mathfrak{su}_2$ for $\lambda \in \mathbb{S}^1$. Decomposing $\alpha_\lambda = \alpha'_\lambda\,dz + \alpha''_\lambda\,d\bar z$ into (1, 0) and (0, 1) parts, we compute

$$\bar\partial\alpha'_\lambda = \frac{1}{2}\begin{pmatrix} u_{z\bar z} & \mathbb{i}\lambda^{-1}u_{\bar z}e^u \\ -\mathbb{i}u_{\bar z}e^{-u} & -u_{z\bar z} \end{pmatrix}, \quad \partial\alpha''_\lambda = \frac{1}{2}\begin{pmatrix} -u_{z\bar z} & -\mathbb{i}u_z e^{-u} \\ \mathbb{i}\lambda u_z e^u & u_{z\bar z} \end{pmatrix},$$

$$\left[\alpha'_\lambda,\, \alpha''_\lambda\right] = \frac{1}{4}\begin{pmatrix} -e^{2u} + e^{-2u} & 2\mathbb{i}u_{\bar z}\lambda^{-1}e^u + 2\mathbb{i}u_z e^{-u} \\ -2\mathbb{i}\lambda u_z e^u - 2\mathbb{i}u_{\bar z}e^{-u} & e^{2u} - e^{-2u} \end{pmatrix}.$$

Now $2\,d\alpha_\lambda + [\,\alpha_\lambda \wedge \alpha_\lambda\,] = 0$ is equivalent to $\bar\partial\alpha'_\lambda - \partial\alpha''_\lambda = [\alpha'_\lambda,\, \alpha''_\lambda]$, which holds if and only if u solves the sinh-Gordon equation

$$\partial\bar\partial\,2u + \sinh(2u) = 0. \tag{1.3}$$

For a smooth solution u of the sinh-Gordon equation, we can integrate the initial value problem

$$\begin{cases} dF_\lambda = F_\lambda\,\alpha_\lambda \\ F_\lambda(0) = \mathbb{1} \end{cases} \tag{1.4}$$

to obtain a map $F_\lambda : \mathbb{C} \times \mathbb{C}^\times \to \mathrm{SL}(\mathbb{C})$, which is called an extended frame for u. Note that $F_\lambda \in \mathrm{SU}_2$ for $\lambda \in \mathbb{S}^1$. For distinct $\lambda_1,\ \lambda_2 \in \mathbb{S}^1$, define the map $f : \mathbb{C} \to \mathrm{SU}_2 \cong \mathbb{S}^3$ by

$$f = F_{\lambda_1}F_{\lambda_2}^{-1}. \tag{1.5}$$

We refer to the two distinct unimodular numbers $\lambda_1,\ \lambda_2$ as the sym points. Straightforward computations reveal that f is a conformal immersion with constant mean curvature

$$H = \mathbb{i}\,\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}. \tag{1.6}$$

The induced metric $f_*\langle\cdot,\ \cdot\rangle = v^2 dz \otimes d\bar z$ has conformal factor

$$v^2 = \frac{e^{2u}}{H^2 + 1} \tag{1.7}$$

and the immersion has constant Hopf differential $Q\,dz^2$ with

$$Q = \frac{\mathbb{i}\,(\lambda_1^{-1} - \lambda_2^{-1})}{4}. \tag{1.8}$$

We in fact get an $\mathbb{S}^1$-family of isometric conformal CMC immersions, called an associated family, which is obtained by simultaneously rotating $\lambda_1,\ \lambda_2$ while keeping the angle between them fixed. Thus each member of an associated family has the same mean curvature, but different Hopf differential.

Let F_λ be an extended frame for a CMC immersion $f : \mathbb{C} \to \mathbb{S}^3$, so that $f = F_{\lambda_1}F_{\lambda_2}^{-1}$ for two distinct unimodular numbers $\lambda_1,\ \lambda_2$. Suppose we have a lattice

$$\Gamma = \gamma_1\,\mathbb{Z} \oplus \gamma_2\,\mathbb{Z}.$$

Periodicity $f(z + \gamma_j) = f(z)$ in terms of the extended frame then reads

$$F_{\lambda_1}(\gamma_j) = F_{\lambda_2}(\gamma_j) = \pm\mathbb{1}. \tag{1.9}$$

Homogeneous tori have constant mean curvature, and up to isometry there is exactly one homogeneous torus for each value of the mean curvature $H \in (-\infty,\ \infty)$. The next proposition singles out a simple representative for each value of the mean curvature.

Proposition 1.1. *Let $\lambda = \exp(\mathbb{i}t),\ t \in (0,\ \pi)$. The map $f : \mathbb{C}/\Gamma \to \mathbb{S}^3$, $f = F_\lambda\,F_{1/\lambda}^{-1}$ with*

$$F_\lambda = \begin{pmatrix} \cos\mu_\lambda & \mathbb{i}\,\lambda^{-1/2}\sin\mu_\lambda \\ \mathbb{i}\,\lambda^{1/2}\sin\mu_\lambda & \cos\mu_\lambda \end{pmatrix} \tag{1.10}$$

and

$$\mu_\lambda = \mu_\lambda(z) = \frac{\pi}{2}\left(z\,\lambda^{-1/2} + \bar z\,\lambda^{1/2}\right) \tag{1.11}$$

is a homogeneous torus. Its period lattice is generated by

$$\gamma_1(t) = \pi\,\sec(t/2), \qquad \gamma_2(t) = \pi\,\mathbb{i}\,\csc(t/2) \tag{1.12}$$

and it has mean curvature $H(t) = -\cot(t)$.